UNIT-5 Graph Theory and Combinatorics

Graphs

Def:- A Graph G is an algebraic (V, E, ψ) in which V is a non-empty set of vertices (nodes), E is the Set of edges and ψ is a mapping from the set E to the set of unordered or ordered pairs of elements of V.

Vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$

Edge set $E(G) = \{(u, v) : u, v \in V(G)\}$

Graph is denoted by $G = (V, E)$ or $G(V, E)$.

Basic Terminology:→

1. **Incident edge:** An edge $e \in E$ that joins the vertices u and v is said to be incident on each of its end Points u and v.

2. **Adjacent vertices:** Any Pair of vertices that is connected by an edge in a graph is called Adjacent vertices.

3. **Isolated Vertex:** A vertex that is not adjacent to another vertex is called an isolated vertex.

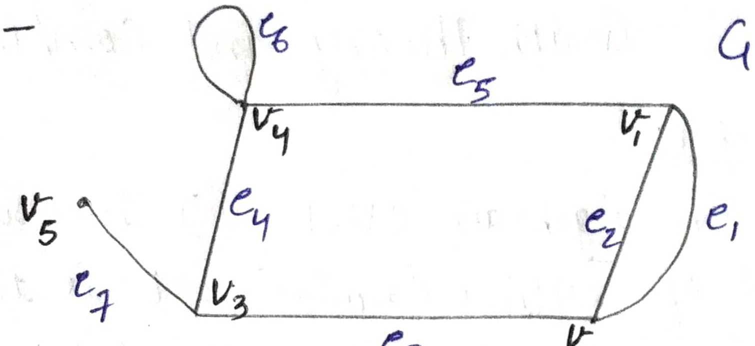
4. **Finite and Infinite Graph:** A Graph $G(V, E)$ is said to be finite if It has a finite number of vertices and finite number of edges. otherwise, It is an Infinite Graph.

5. **order of a Graph:** If G is a finite Graph, Then the number of vertices in G is called order of G, denoted by $|V(G)|$. or n

6. **Size of a Graph:** In a finite Graph $G(V, E)$, The no. of edges is called size of G, denoted by $|E(G)|$, or m.

→ A Graph of n order and m size is referred as (n, m)

Example :-



G(V, E)

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ $\quad |E| = 7$

$V = \{v_1, v_2, v_3, v_4, v_5\}$ $\quad |V| = 5$

So order of G = 5

size of G = 7

(7.) **Self loop | loop:** An edge of a Graph that joins a vertex to itself is called a self loop or simply loop.

It is denoted by (v_i, v_i).

e.g → In above graph G, e_6 is self loop.

(8.) **Multiple or Parallel edges:**

When two or more edges associated with a given pair of vertices, Then such edges are called multiple edges or Parallel edges.

Two edges (v_i, v_j) and (v_k, v_l) are Parallel if

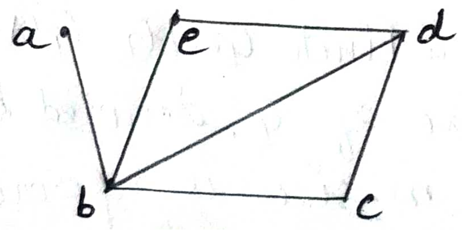
$v_i = v_k$ and $v_j = v_l$

e.g → In above Graph G, e_1 and e_2 are Parallel edges.

(#) **Undirected and Directed Graph:**

→ A Graph G(V, E) is called an undirected graph if each edge e_k is associated with an unordered pair (v_i, v_j) of vertices. i.e a Graph in which edges have no direction.

Ex:-

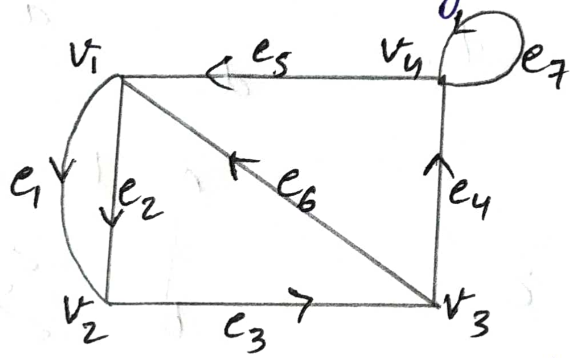


unordered pair means

$(v_i, v_j) = (v_j, v_i)$

→ A Graph $G(V, E)$ is called Directed Graph if each edge e_K is associated with ordered pair (V_i, V_j) of vertices. i.e a Graph in which each edge has a direction.

Ex:-



Note:- If $e = (V_i, V_j)$ is a directed edge in a Digraph or Directed graph, Then

(1) V_i is called Initial vertex of e and V_j is called terminal vertex of e

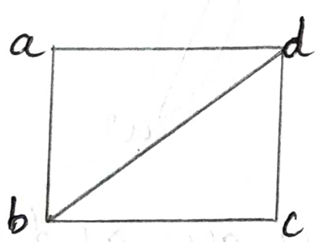
(2) e is said to be incident from V_i and to be incident to V_j.

(3) V_i is adjacent to V_j, and V_j is adjacent from V_i.

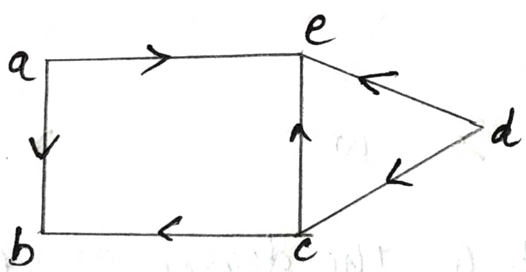
Types of Graph: →

(1) Simple Graph: A Graph which has neither self loops nor multiple edges is called a Simple Graph.

e.g →



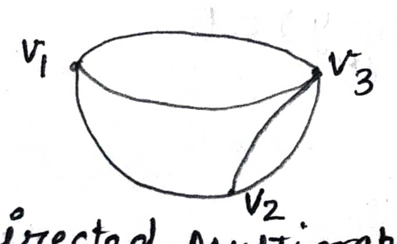
Undirected simple Graph



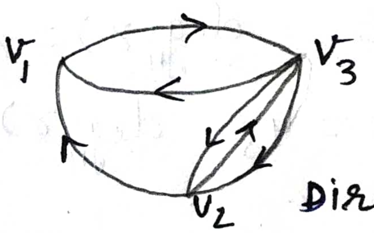
Directed simple graph

(2) Multigraph:- Any Graph which contains some multiple edges is called a Multigraph. In Multigraph, no loops are allowed.

e.g →



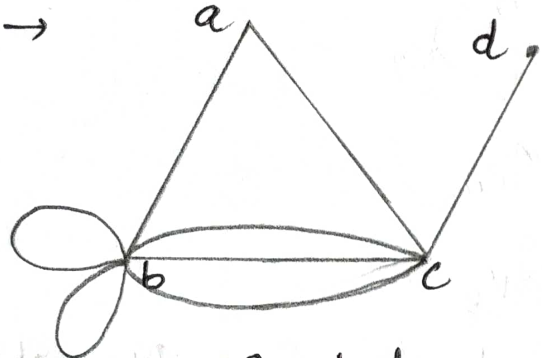
Undirected Multigraph



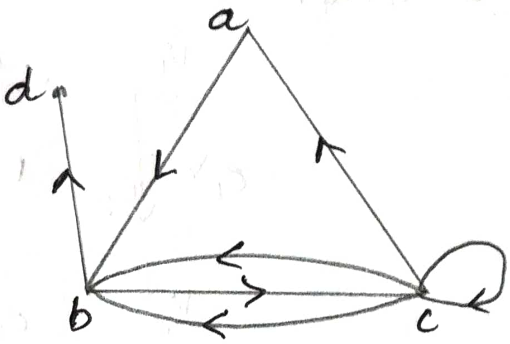
Directed Multigraph.

3. **Pseudograph:** A Graph with self loops and multiple edges is called a Pseudograph.

e.g →



Undirected Pseudograph



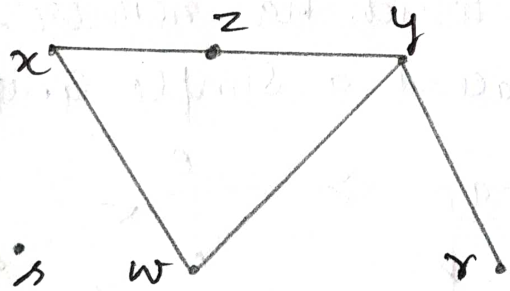
Directed Pseudograph.

Degree of a Vertex: →

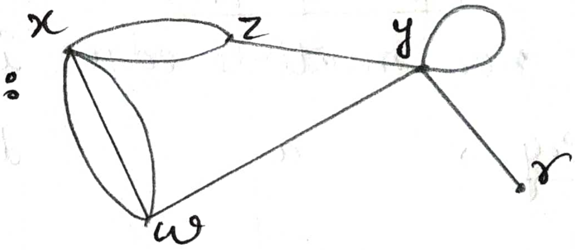
The degree of a vertex of an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of a vertex v in a Graph G is denoted by $\deg_G(v)$.

Example:-

G:



H:



In Graph G, the degree of vertices are as follows:

$\deg_G(x) = 2 = \deg_G(z) = \deg_G(w)$

$\deg_G(y) = 3$, $\deg_G(r) = 1$, $\deg_G(s) = 0$

In Graph H, the degree of vertices are as follows:

$\deg_H(x) = 5$, $\deg_H(z) = 3$, $\deg_H(y) = 5$

$\deg_H(w) = 4$, $\deg_H(r) = 1$

→ A vertex of degree 0 is called Isolated vertex.

→ A vertex of degree 1 is called Pendant.

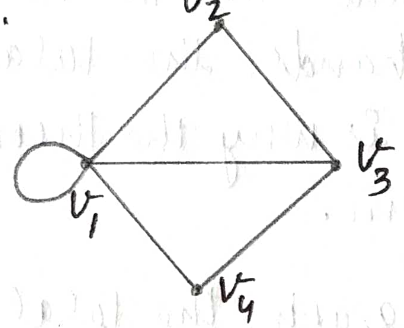
→ A vertex of a Graph is called odd vertex if its degree is odd and a vertex of even degree is called even vertex

Degree Sequence of a Graph: →

If $v_1, v_2, \dots, v_n$ are the n vertices of a Graph G and let $d_1, d_2, \dots, d_n$ be their respective degrees. If the sequence $(d_1, d_2, \dots, d_n)$ is mondonically Increasing. Then it is called the degree sequence of the Graph G.

e.g →

G:



Here, $deg(v_1) = 5$, $deg(v_2) = 2$, $deg(v_3) = 3$, $deg(v_4) = 2$

Since $deg(v_2) \le deg(v_4) \le deg(v_3) \le deg(v_1)$

i.e $2 \le 2 \le 3 \le 5$

Therefore, the degree sequence of G is $(2, 2, 3, 5)$.

Theorem:- A simple graph with atleast two vertices has at least two vertices of same degree.

Imp

Handshaking theorem:-

If $G = (V, E)$ be an undirected graph with e edges.

Then $$\boxed{\sum_{v \in V} deg_G(v) = 2e}$$

i.e the sum of degrees of the vertices in an undirected graph is even.

Proof:- Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degrees counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end. Thus the sum of the degrees is equal to twice the number of edges in the Graph.

Note:- Handshaking Theorem applies even if multiple edges and loops are present. This Thm holds Rule that if several People shake hands. the total no. of hands shake must be even that is why the theorem is called handshaking theorem.

Imp

Rt:- In a non directed graph, the total number of odd degree vertices is even.

Proof:- Let $G=(V,E)$ be a non directed graph (Undirected). Let U denotes the set of even degree vertices and W denotes the set of odd degree vertices in G.

Then $$\sum_{v_i \in V} deg_G(v_i) = \sum_{v_i \in U} deg_G(v_i) + \sum_{v_i \in W} deg_G(v_i)$$

$$\Rightarrow \quad 2e - \sum_{v_i \in U} deg_G(v_i) = \sum_{v_i \in W} deg_G(v_i) \quad \text{—} ①$$

Now $\sum_{v_i \in U} deg_G(v_i)$ is even as the sum of degrees of even degree vertices is always even.

Therefore, from ① $\sum_{v_i \in W} deg_G(v_i)$ is even

∴ Since for each $v_i \in W$, $deg_G(v_i)$ is odd,

The number of odd vertices in G must be even.

(#) In degree and out degree :—

In a directed Graph G, the outdegree of a vertex v of G, denoted by $deg_G^+(v)$, is the number of edges beginning from v and the Indegree of v, denoted by $deg^-(v)$, is the number of edges ending at v.

→ The Sum of the Indegree and outdegree of a vertex is Called the total degree of the vertex.

→ A vertex of zero Indegree is called Source and a vertex of zero outdegree is called a Sink.

Theorem:- Let $G = (V, E)$ be a directed graph with e edges

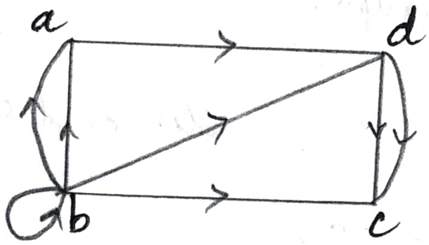
Then $$\boxed{\sum_{v\in V} deg_G^+(v) = \sum_{v\in V} deg_G^-(v) = e}$$

i.e the sum of outdegrees of the vertices of a digraph G equals the sum of Indegrees of the vertices which equals the number of edges in G.

Proof:- Any directed edge (u, v) contributes 1 to the Indegree of v and 1 to the outdegree of u, further a loop at v contributes 1 to the Indegree of v and 1 to the outdegree of v. Hence Proved.

e.g →

G:



Here $d^-(a) = 2$ $\quad d^-(b) = 1$ $\quad d^-(c) = 3$ $\quad d^-(d) = 2$

$d^+(a) = 1$ $\quad d^+(b) = 5$ $\quad d^+(c) = 0$ $\quad d^+(d) = 2$

So $d^-(a) + d^-(b) + d^-(c) + d^-(d) = 2+1+3+2 = 8 = e$

& $d^+(a) + d^+(b) + d^+(c) + d^+(d) = 1+5+0+2 = 8 = e$

no. of edges in $G = e = 8$

Th^m:- The degree of a vertex of a simple Graph G on n vertices can not exceed $n-1$.

i.e In a simple graph G, $\deg(v) \leq n-1$, $\forall v \in V(G)$.

Proof:- Let v be a vertex of a simple Graph G.
Since G is simple, so no multiple edges or self loops are allowed in G.
Thus, v can be adjacent to atmost all the Remaining $n-1$ vertices of G.
Hence, v may have maximum degree $n-1$ in G.
Then $0 \leq \deg_G(v) \leq n-1$, $\forall v \in V(G)$

Imp

Theorem:- The maximum number of edges in a simple graph with n vertices is $\frac{n\cdot(n-1)}{2}$.

OR A simple Graph G with n vertices may have atmost $\frac{n\cdot(n-1)}{2}$ edges.

Proof:- By Handshaking theorem.

$$\sum_{i=1}^{n} d(v_i) = 2e$$

where e is the number of edges with n vertices in the Graph G

$$\Rightarrow \quad d(v_1) + d(v_2) + \cdots + d(v_n) = 2e \quad \text{—(1)}$$

Since the maximum degree of each vertex in G can be $n-1$ as G is simple Graph.

So (1), Reduces to

$$(n-1) + (n-1) + \cdots \text{ to } n \text{ terms} = 2e$$

$$\Rightarrow \quad n\cdot(n-1) = 2e$$

$$\Rightarrow \quad \boxed{e = \frac{n\cdot(n-1)}{2}}$$

Hence the maximum number of edges in any simple Graph with n vertices is $\frac{n\cdot(n-1)}{2}$

Ques. 1 Show that there does not exists any graph of order 5 with the degree sequence (4, 3, 3, 2, 1).

Proof:- Since the sum of degrees of vertices

$$4+3+3+2+1 = 13 \text{ which is odd}$$

So there exist no graph corresponding to this degree sequence.

Ques. 2 Is there a simple graph corresponding to the following degree sequences?

i.) (2, 2, 4, 6) (ii.) (1, 1, 1, 1)

Soln:- i.) No. of vertices in the graph is 4 and the maximum degree of a vertex is 6 which is not possible as in a simple graph the maximum degree can not exceed one less than the no. of vertices.

So $\nexists$ any graph (simple) of degree sequence (2, 2, 4, 6)

ii.) The sum of the degrees of all vertices

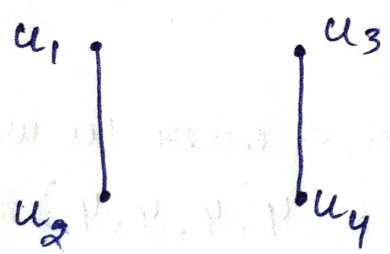
$$1+1+1+1 = 4 \text{ which is even.}$$

The no. of odd vertices is 4, even

Hence a simple disconnected graph is possible which has 4 vertices of degree 1 each.

Maximum no. of edges can be $\frac{4\cdot(4-1)}{2} = 2\cdot 3 = 6$

but we have no. of edges $\frac{4}{2} = 2 < 6$



Ques.3 Does there exists a graph of order 6 with the degree Sequence (3, 3, 3, 3, 3, 3) ?

Sol^n:- The sum of the degrees of all vertices is

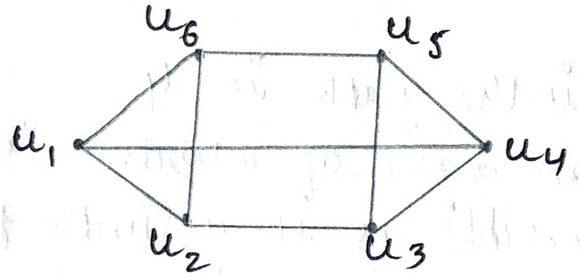
$3+3+3+3+3+3 = 18$, even

The no. of odd vertices is 6, even

The maximum degree 3 does not exceed $6-1=5$

The maximum no. of edges of 6 vertices $\frac{6\cdot(6-1)}{2} = 15$

So a simple graph exists of 6 vertices & 9 edges.



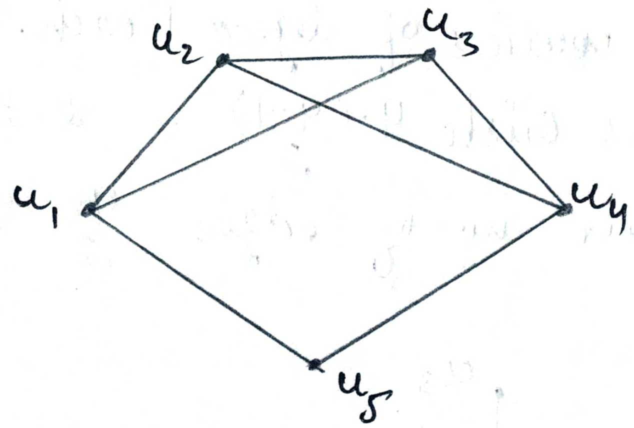
Ques.4. Does there exists a graph of order 5 with degree sequence (3, 3, 3, 3, 2) ?

Sol^n:- The sum of the degrees is 14, even

The number of odd vertices is 4, even

maximum degree is 3 does not exceed $5-1=4$

So ∃ a simple graph of 5 vertices & 7 edges



Ques.5 Does there exists a graph of order 10 with the degree sequence (5, 5, 5, 5, 5, 5, 4, 4, 4, 4) ?

Havel - Hakimi theorem :-

A finite sequence $(d_1, d_2, \ldots, d_{n-1})$ is graphical iff the sequence $(d_1 - 1, d_2 - 1, \ldots, d_k - 1, d_{k+1}, \ldots, d_{n-1})$ is also graphical.

where $(d_1, d_2, \ldots, d_n)$ is a non-increasing sequence of degrees of vertices of a graph G.

* A finite sequence $(d_1, d_2, \ldots, d_{n-1})$ of non-negative Integers is called graphical if there is a simple graph with this degree sequence.

Ex:- check which of the following sequences are graphical

i.) (3, 3, 3, 3, 2) ii.) (3, 3, 3, 3, 3, 3) iii) (5, 5, 4, 4, 4, 3, 3)

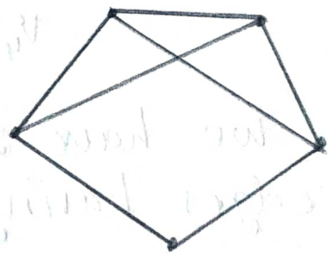
Sol^n:- i.) (3, 3, 3, 3, 2)

x (2 2 2 2)

x (1 1 2)

(2 1 1)

x (0 0)



⇒ ∃ a simple graph corresp. to given degree sequence

⇒ Degree sequence is graphical

iii.) (5, 5, 4, 4, 4, 3, 3)

x (4 3 3 4 3 3)

(4 4 3 3 3 3)

x (3 2 2 3 3)

(3 3 3 2 2)

x (2 2 1 2)

(2 2 2 1)

x (1 1 1)

x (0 1)

(1 0)

x (-1)

The degree seq. does not vanish so ∄ any simple graph

⇒ The degree seq. is not graphical

Types of Graphs :→

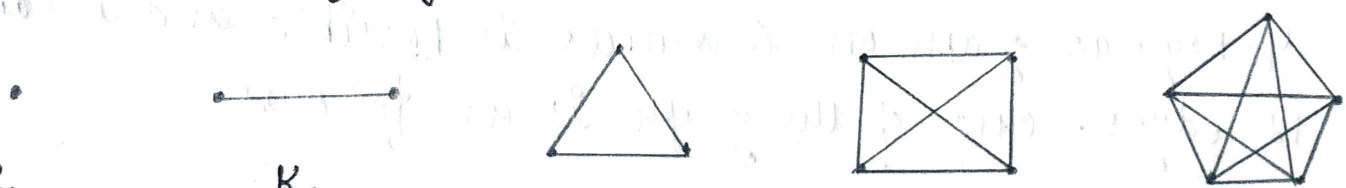
1. **Null Graph :–** A graph which contains only isolated vertex is called a null graph i.e the set of edges in a null graph is empty. Null graph on n vertices is denoted by N_n

e.g →

N_4 N_5

2. **Complete Graph :–** A Simple Graph G is said to be complete if every vertex in G is connected with every other vertex i.e if G contains exactly one edge between each pair of distinct vertices. A complete Graph is denoted by K_n with n vertices.

$$\text{No. of edges in } K_n = \frac{n.(n-1)}{2}$$



K_1 K_2 K_3 K_4 K_5

3. **Regular Graph :–** A Graph in which all vertices are of equal degree is called a regular graph. if the degree of each vertex is r, then the graph is called a regular graph of degree r.

Note :–

→ Every Null graph is regular of degree zero.

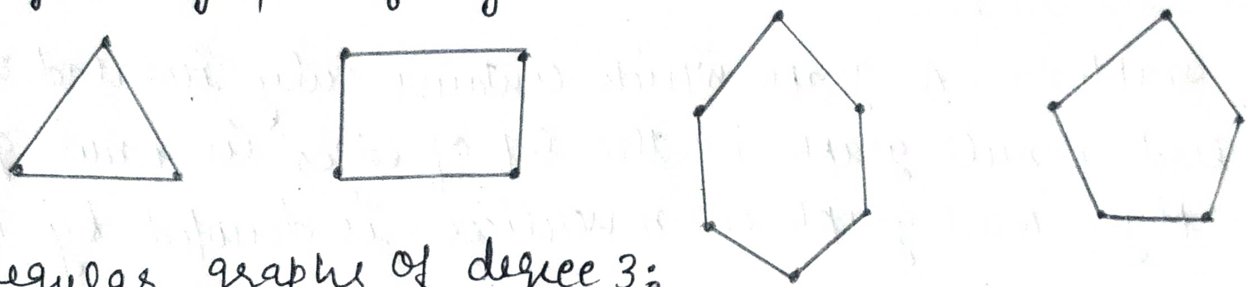
→ The complete graph K_n is regular of degree $n-1$.

→ If G has n vertices and is regular of degree r, Then G has $\frac{r.n}{2}$ edges.

→ A Regular graph need not be complete

e.g  is a Regular graph but not complete.

Regular graphs of degree 2 :



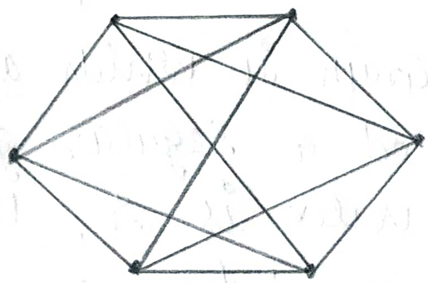
Regular graphs of degree 3:

<u>Ques.</u> Does there exists a 4 - regular graph on 6 - vertices? if so construct a graph.

<u>Sol^n:-</u> No. of edges in r-regular graph on n vertices

$$q = \frac{r.n}{2} = \frac{4 \times 6}{2} = 12$$

4-Regular graph on 6 vertices is Possible and it Contains 12 edges. one of the graph is as follows:



Note:- Every 4- Regular graph contains 3 - regular graph.

<u>Ques.</u> what is the size of r- regular (p, q) graph?

No. of vertices   No. of edges

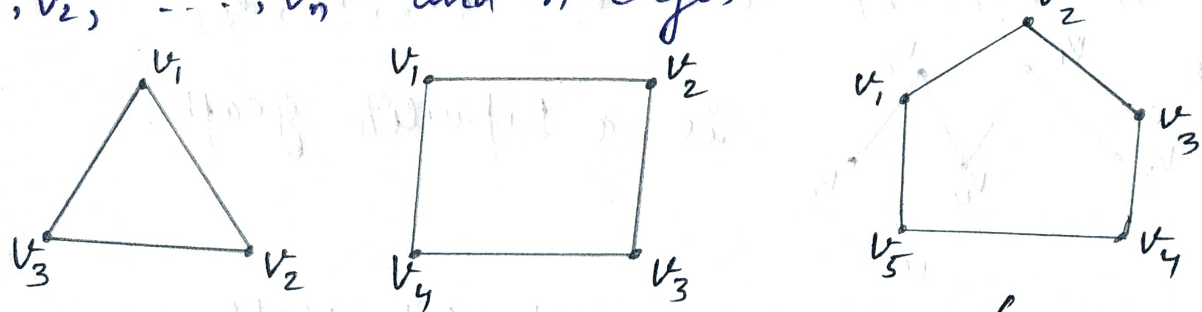
<u>Sol^n:-</u> In r- regular graph $\deg_G(v_i) = r$, $\forall\ v_i \in V(G)$

By Handshaking theorem, $2q = \sum \deg_G(v_i)$

$$2q = \sum r = p \times r$$

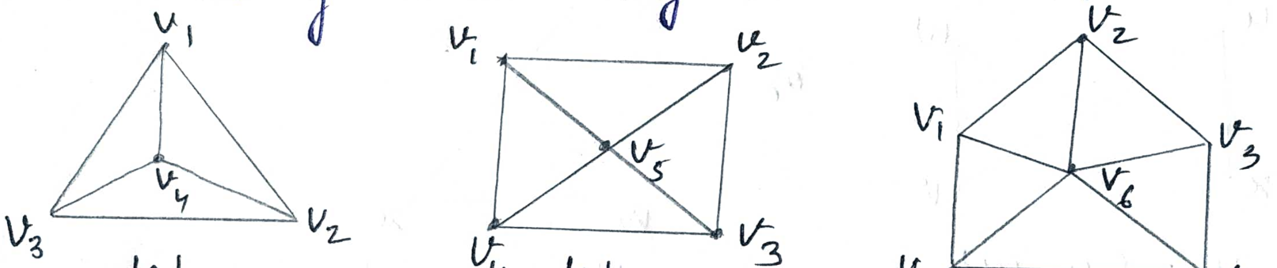
$$\Rightarrow \quad q = \frac{p \times r}{2}$$

④ <u>Cycle Graph</u>:— The cycle graph C_n ($n \geq 3$), of length n is a connected graph which consists of n vertices $v_1, v_2, \ldots, v_n$ and n edges.



C_3 C_4 C_5

⑤ <u>Wheel Graph</u>:— The wheel graph W_n ($n > 3$) is obtained from C_{n-1} by adding a vertex v inside C_{n-1} and connecting it to every vertex in C_{n-1}.



W_4 W_5 W_6

<u>Note</u>:- ① C_n is a regular graph of degree 2

② W_n is a regular graph for $n = 4$

③ W_n has n vertices and $2n-2$ edges.

Imp

\# <u>Bipartite Graph</u>: →

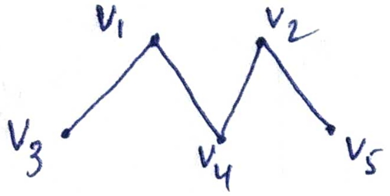
A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into two subsets (disjoint) V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex in V_2 so that no edge in G connects either two vertices in V_1 or two vertices in V_2.

(V_1, V_2) is called Bipartition of G.

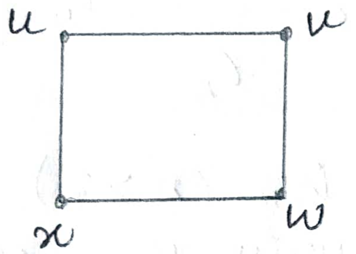
No loops are allowed in Bipartite graph

Example :-

① Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ such that
$V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, v_5\}$

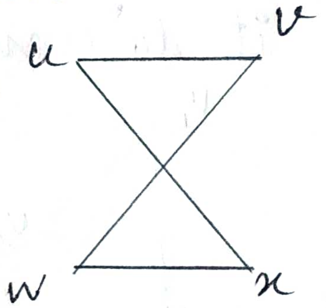
Then


is a Bipartite graph.

②


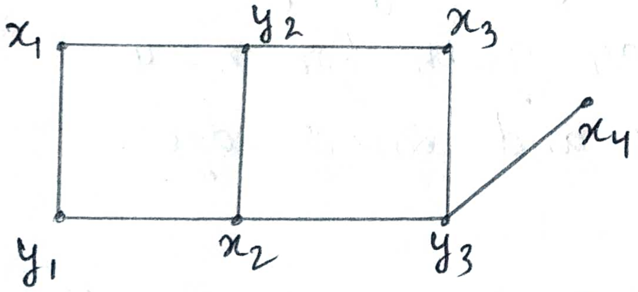
is a Bipartite graph

as it can be Redrawn as

or


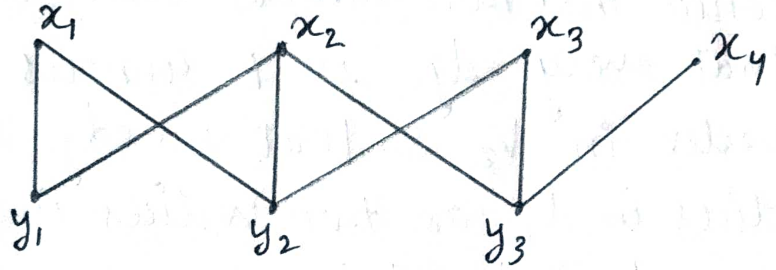
$V_1 = \{u, w\}$
$V_2 = \{x, v\}$

$V_1 = \{u, w\}$
$V_2 = \{v, x\}$

③


is a Bipartite graph

as it can be Redrawn as

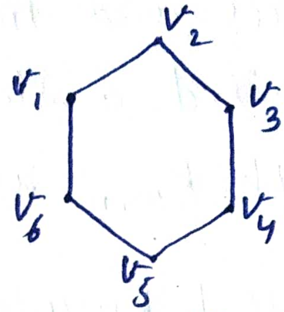


$V_1 = \{x_1, x_2, x_3, x_4\}$
$V_2 = \{y_1, y_2, y_3\}$.

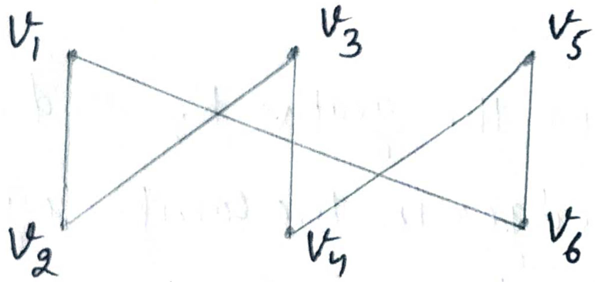
Example: Show that C_6 is a Bipartite graph.

Soln:-

$C_6 \longrightarrow$



C_6 is a Bipartite graph since its vertex set $V = \{v_1, v_2, \ldots v_6\}$ can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.



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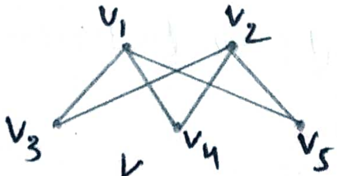
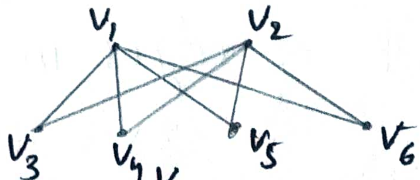
complete Bipartite graph : →

A Bipartite graph $G = (V, E)$ is said to be complete Bipartite if each vertex of V_1 is connected to each vertex of V_2 where V_1 and V_2 are the two distinct partitions of the vertex set V. complete Bipartite graph G is denoted by $K_{m,n}$, where m and n are the number of vertices in vertex sets V_1 and V_2 respectively.

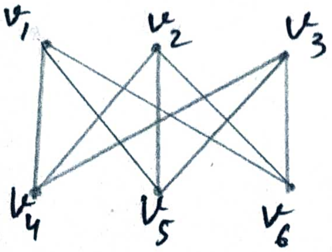
e.g →



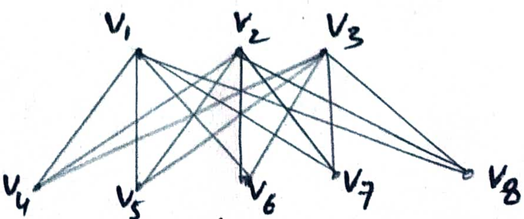
$K_{2,3}$



$K_{2,4}$



$K_{3,3}$



$K_{3,5}$

Note:-

1. $K_{m,n}$ has $m+n$ vertices and $m \cdot n$ edges.
2. Any graph $K_{1,n}$ is called a star graph.
3. A complete Bipartite graph $K_{m,n}$ is not a regular graph if $m \neq n$.
4. K_5 and $K_{3,3}$ are called Kuratowski graphs.
5. A graph which contains a triangle can not be bipartite
6. No. of edges in a Bipartite graph with n vertices is atmost $\frac{n^2}{4}$.

(2018)
Ques.1 How many edges the graphs K_7 and $K_{3,6}$ have?

Solⁿ:- The Number of edges in the complete graph

$$K_7 = \frac{7 \cdot (7-1)}{2} = \frac{7 \cdot 6}{2} = 7 \cdot 3 = 21$$

the Number of edges in complete Bipartite graph

$$K_{3,6} = 3 \times 6 = 18$$

(2022)
Ques.2 How many edges the graphs K_7 and $K_{3,3}$ have?

Solⁿ:- The Number of edges in complete graph

$$K_7 = \frac{7 \cdot (7-1)}{2} = \frac{7 \cdot 6}{2} = 7 \cdot 3 = 21$$

The Number of edges in complete Bipartite graph $K_{3,3} = 3 \times 3 = 9$.

Representation of Graphs :→

(I.) Matrix Representation :-

Matrix are commonly used to Represent graphs for computer Processing. The advantages of representing graphs in matrix form lies on the fact that many results of matrix algebra can be readily applied to study the structural Properties of graphs from an algebraic Point of view.

There are two matrix Representation of graphs Adjacency matrix and Incidence matrix.

(1.) Adjacency Matrix :-

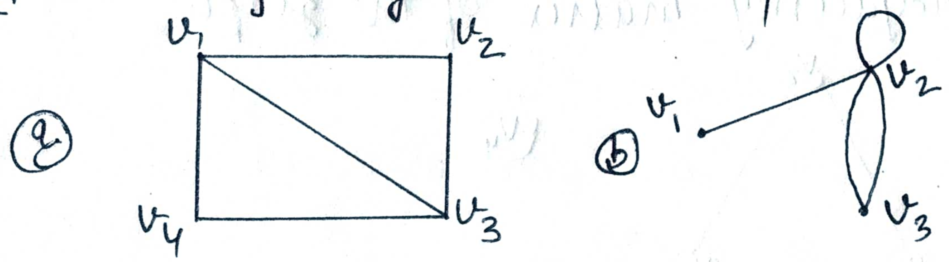
@ For Undirected graph →

The adjacency of an undirected graph G with n vertices and no Parallel edges is an $n \times n$ matrix $A = [a_{ij}]_{n \times n}$ whose elements are given by

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge b/w ith \& jth vertices} \\ 0 & \text{if there is no edge b/w them} \end{cases}$$

Note:- A is Symmetric matrix i.e $a_{ij} = a_{ji}$ for all i and j.

Ques.1 Use Adjacency matrix to represent the graphs



Sol^n:- (a) Since there are four vertices v_1, v_2, v_3, v_4

So Adjacency matrix representing the graph will be a Square Matrix of order four. So Required Adjacency Matrix A

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

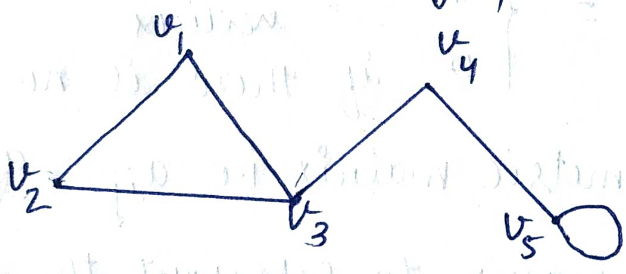
(b) we order the vertices as v_1, v_2 and v_3. So Adjacency matrix representing the graph with loop and Multiple edges is given by

$$A = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$

Ques.2 Draw the Undirected graph represented by Adjacency matrix A given by

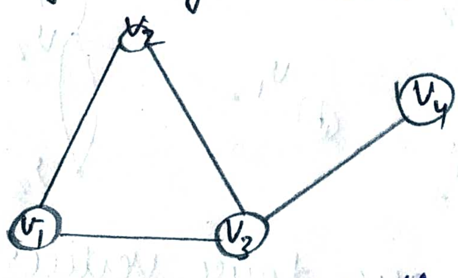
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Sol^n:- Since the given matrix is a square matrix of order 5 the graph G has 5 vertices v_1, v_2, v_3, v_4 and v_5. The Required undirected graph is



2023

Ques 3. Draw an Adjacency matrix of the graph



Sol^n:- we order the vertices as v_1, v_2, v_3 and v_4. The the Adjacency matrix of given Graph is

$$A = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b) for Directed graph →

The Adjacency matrix of a Digraph D, with n vertices is the matrix $A = [a_{ij}]_{n \times n}$ in which

$$a_{ij} = \begin{cases} 1, & \text{if arc } (v_i, v_j) \text{ is in } D \\ 0, & \text{otherwise} \end{cases}$$

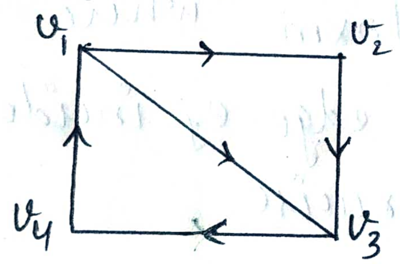
Note:–

(1) A is not necessary symmetric, since there may not be an edge from v_i to v_j when there is an edge from v_j to v_i.

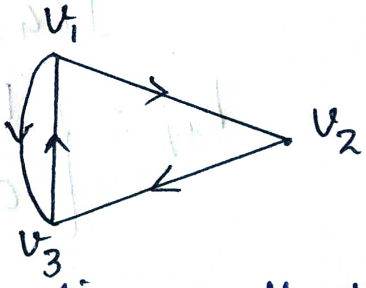
(2) The Adjacency matrices can also be used to Represent directed multigraphs. Such matrices are not zero-one matrices when there are multiple edges in the same direction connecting two vertices. In the Adjacency matrix for a directed multigraph, a_{ij} equals the no. of edges that are associated to (v_i, v_j).

Ques.1 Use Adjacency Matrix to represent the graphs

(a)



(b)



Solⁿ:– (a) Taking the order of the vertices as v_1, v_2, v_3 and v_4 Then the Adjacency matrix representing the Digraph is given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

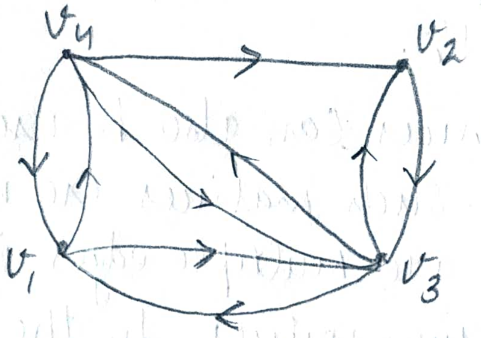
(b)

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Ques.2. Draw the Digraph G corresponding to Adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solⁿ:- Since the given matrix is a square matrix of order four, the graph G has 4 vertices v_1, v_2, v_3 and v_4. Draw an edge from v_i to v_j where $a_{ij} = 1$. The Required graph is



(2.) Incidence Matrix :→

(a) for Undirected graph →

Incidence matrix of an undirected graph $G = (V, E)$ with n vertices and m edges all labelled is $n \times m$ matrix

$$I(G) = [b_{ij}]_{n \times m} \quad \text{where}$$

$$b_{ij} = \begin{cases} 1, & \text{when edge } e_j \text{ incident with } v_i \\ 0, & \text{otherwise} \end{cases}$$

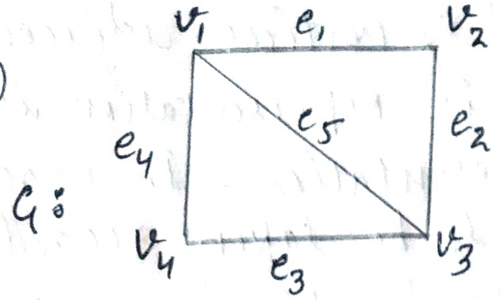
(b) For Directed graph →

The Incidence matrix $I(D) = [b_{ij}]$ of digraph D with n vertices and m edges is the $n \times m$ matrix in which

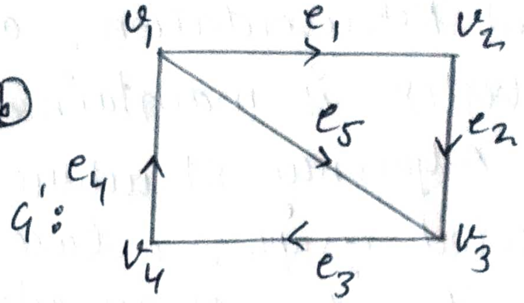
$$b_{ij} = \begin{cases} 1, & \text{if arc } j \text{ is directed away from } v_i \\ -1, & \text{if arc } j \text{ is directed towards } v_i \\ 0, & \text{otherwise} \end{cases}$$

Ques. 1 Find the incidence matrix to Represent the graphs

(a)



(b)



Solⁿ:- (a) The Incidence matrix of G is

$$
I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}
$$

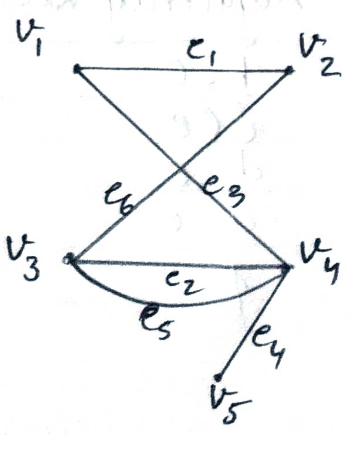
(b) The Incidence matrix of Digraph G' is

$$
I(D) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}
$$

Ques. 2. Draw the graph whose Incidence matrix is

$$
\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
$$

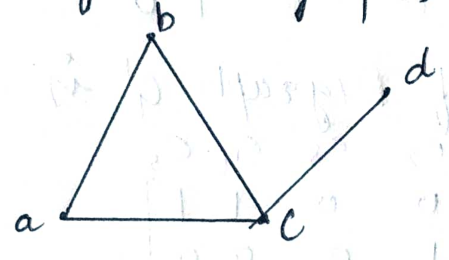
Solⁿ:- Since the given matrix has 5 rows and 6 coloums So its corresponding graph has 5 vertices and 6 edges The graph of the given incidence matrix is

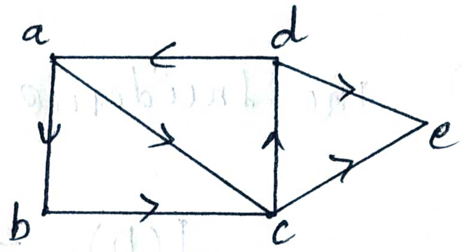


(II.) Linked Representation :→

In Linked Representation, a list of vertices adjacent to each vertex is maintained. This Representation is also called Adjacency structure representation. In case of a directed graph, a case has to be taken according to the Direction of an edge, while placing a vertex in the Adjacent structure representation of another vertex. The Linked Representation is described by means of two examples.

Example:- write adjacency structure / Linked Representatⁿ for the graphs.

G: 

G': 

Solⁿ:- The Adjacency Structure is given in table for G

vertex	Adjacency list
a	b, c
b	a, c
c	a, b, d
d	c
e	ϕ

The Adjacency structure representation is given in table for directed graph G'

vertex	Adjacency list
a	b, c
b	c
c	d, e
d	a, e
e	ϕ

Some Basic Terminology :→

Walk:- A walk in a Graph G is a finite alternating sequence
$$v_0 - e_1 - v_1 - e_2 - v_2 - e_3 - \dots - e_n - v_n$$
of vertices and edges of the graph such that each edge e_i joins vertices v_{i-1} and v_i, $1 \le i \le n$.

→ The end vertices v_0 and v_n are called the end or terminal vertices of walk.

→ The vertices $v_1, v_2, \dots, v_{n-1}$ are called its Internal Vertices

→ length of the walk = no. of edges in a walk.

→ A walk in which terminal vertices are distinct is called open walk.

→ A walk in which terminal vertices are same is called closed walk.

Note:- A walk may Repeat both vertices and edges.

Trail:- A walk is called a trail if all its edges are distinct.

Path:- A walk is called a Path if all its vertices and edges are distinct.

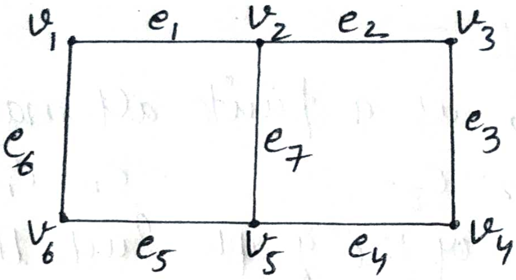
Circuit:- A closed trail is called a circuit

cycle:- A Path in which only repeated vertex is the first vertex is called a cycle to describe such a closed Path.

Particularly,

	Repeated Edge	Repeated vertex	Terminal vertices are same ?
open walk	allowed	allowed	No
closed walk	allowed	allowed	yes
trail	Not allowed	allowed	No
circuit	Not allowed	allowed	yes
Path	Not allowed	Not allowed	No
cycle	Not allowed	first and last only	yes

Ex:-



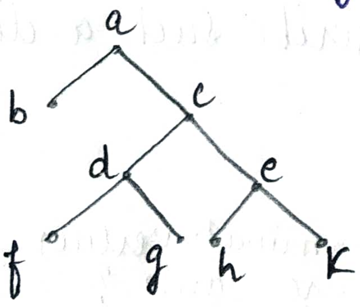
1. The sequence $v_1 - e_1 - v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_7 - v_2 - e_1 - v_1$ is a walk of length 6. It contains repeated vertices v_1, v_2 and repeated edge e_1.
2. The sequence $v_1 - e_1 - v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_7 - v_2$ is a trail. It contains repeated vertex v_2.
3. The sequence $v_1 - e_1 - v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5$ is a Path. It does not contain any repeated vertex and repeated edge.
4. The sequence $v_1 - e_1 - v_2 - e_7 - v_5 - e_5 - v_6 - e_6 - v_1$ is a cycle. It does not contain repeated vertex and repeated edge except the first and last vertex.

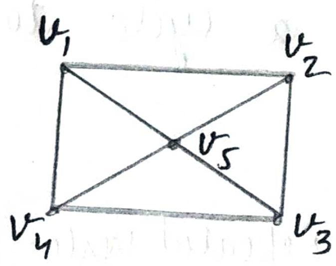
Connected Graphs:-

A Graph is connected if there is a path b/w any two vertices of the graph.

A graph is disconnected if there exist atleast two vertices having no path b/w them.

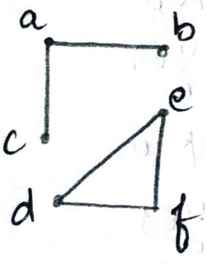
Ex:-

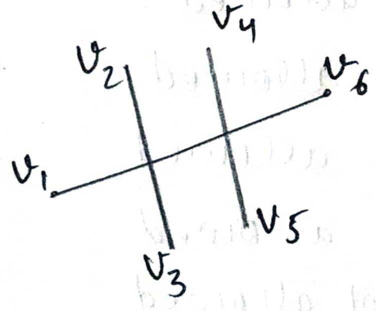




are connected graphs.

whereas





are disconnected graphs.

A Disconnected graph is the union of two or more connected subgraphs each pair of which has no vertex in common. These disjoint connected subgraphs are called connected components of the graph.

A connected component of a Graph is a connected subgraph of largest possible size.

Imp

Theorem:- A simple graph with n vertices and k components can not have more than $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:- Let the no. of vertices in each of the k-components of a graph G be $n_1, n_2, \ldots, n_k$, then we get

$$n_1 + n_2 + \cdots + n_k = n \quad \text{where } n_i \geq 1 \ (i=1,2,\ldots,k)$$

Now

$$\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n-k$$

$$\therefore \left(\sum_{i=1}^{k}(n_i-1)\right)^2 = n^2 + k^2 - 2nk$$

or

$$\sum_{i=1}^{k}(n_i-1)^2 + 2\sum_{i=1}^{k}\sum_{\substack{j=1 \\ i \neq j}}(n_i-1)(n_j-1) = n^2+k^2-2nk$$

or

$$\sum_{i=1}^{k}(n_i-1)^2 + 2\,(\text{non negative terms}) = n^2+k^2-2nk$$

$$[\because n_i - 1 \geq 0,\ n_j - 1 \geq 0]$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 \leq n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \leq n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2n \leq n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - n \leq n^2+k^2-2nk-k+n$$

$$= n(n-k+1) - k(n-k+1)$$

$$= (n-k)(n-k+1) \quad \text{—(1)}$$

we know that the maximum no. of edges in the i^{th} component of G $= {}^{n_i}C_2 = \frac{n_i.(n_i-1)}{2}$

$\therefore$ maximum no. of edges in G is

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - n\right) \le \frac{(n-k)(n-k+1)}{2}$$

$$|E| = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} \le \frac{(n-k)(n-k+1)}{2}$$

Hence Proved

Imp

Euler / Eulerian Graphs →

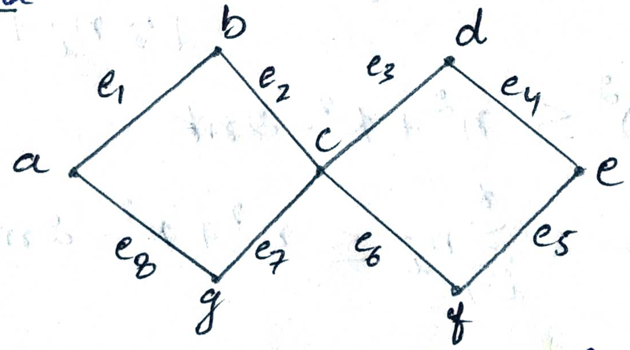
A circuit in a connected graph is called an Euler circuit if it contains every edge of graph exactly once.

A connected Graph with an Euler circuit is called an Euler graph or Eulerian graph.

A Path in a connected graph G is called Euler Path if it contains every edge of the graph exactly once. Since a Path contains every edge exactly once, it is called Euler trail.

Example -

G :



Graph G has an Eulerian circuit $a e_1 b e_2 c e_3 d e_4 e\, e_5 f e_6 c e_7 g e_8 a$.

So it is an Eulerian graph.

Imp.

Necessary and Sufficient condition of Euler graph

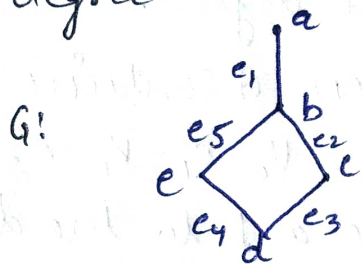
Thm:- A nonempty connected graph G is Eulerian if and only if its vertices are all of even degree.

Proof:- Suppose G is Eulerian. Then G contains an Eulerian circuit, say, from v_0 to v_0 : $v_0 - e_1 - v_1 - e_2 - \dots v_{n-1} - e_n - v_0$. both edges e_1 and e_n contribute a 1 to the degree of v_0. So degree of v_0 is atleast two. Each time the circuit passes through a vertex (including v_0), the degree of the vertex is increased by 2. consequently, the degree of every vertex, including v_0 is an even Integer.

conversely, suppose every vertex of G has even degree. Now we construct an Euler circuit starting at an arbitrary vertex v and going the edges of G exactly once. Since every vertex is of even degree, we can exit from every vertex we enter. if this closed circuit H_1 contains all the edges of G, then G is an Euler graph. if this circuit does not contain all the edges of G the we remove from G all the edges in H_1 and obtain a subgraph $H_2 = G - H_1$ of G formed out of remaining edges. Since both G and H_1 have all their vertices of even degree, the degrees of vertices of H_2 are also even. Since G is connected H_2 and H_1 must have atleast one common vertex say v_1. Starting from v_1 one can construct another new circuit in H_2. This new circuit in H_2 can be combined with H_1 to form a new larger circuit. if it is Eulerian, then G is. if it is not, we continue this procedure to form an Euler circuit. This procedure must terminate since the no. of Edges in G is finite. Thus G contains an Euler circuit & Hence is Eulerian.

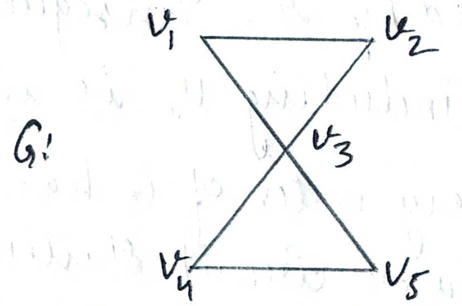
Th^m:- A connected graph contains an Euler trail, but not Euler circuit iff it has exactly two vertices of odd degree.

e.g.-



In Graph G, there are two vertices a and b whose degree is odd. So G has Euler trail $a e_1 b e_2 c e_3 d e_4 e e_5 b$ but not an Euler circuit.

Ex:-1 verify that G has an Eulerian circuit.



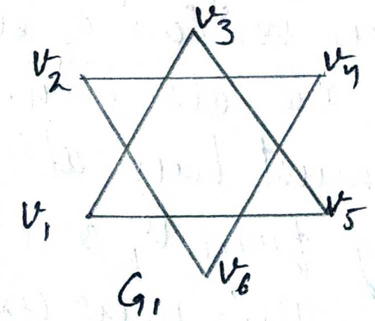
Sol^n:- Here G is connected and all the vertices are having even degree $deg(v_1) = deg(v_2) = deg(v_4) = deg(v_5) = 2$, $deg(v_3) = 4$

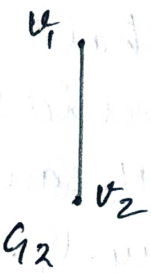
Thus G has a Eulerian circuit

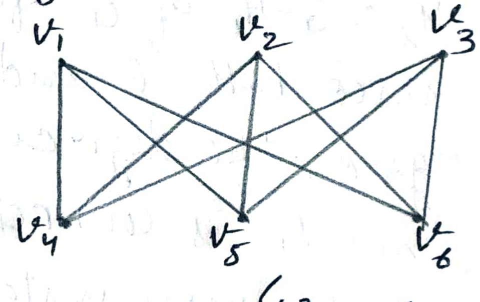
By Inspection, we find the Eulerian circuit

$v_1 - v_3 - v_5 - v_4 - v_3 - v_2 - v_1$

Ex:-2 Show that the Graphs G_1, G_2, G_3 contain no Euler circuit







Sol^n:- (1) Graph G_1 does not contain Euler circuit, since it is not Connected

(2) Graph G_2 is connected, but vertices v_1 and v_2 are of degree 1. Hence it does not contain Euler circuit

(3) All the vertices of the Graph G_3 are of degree 3, Hence it does not contain Euler circuit.

Ques.1 For what values of n, the complete graph K_n is Eulerian?

Solⁿ:- K_n is a complete graph of n vertices in which degree of each vertex is $n-1$.
Since a graph is Eulerian iff it is connected and degree of each vertex is even
So we conclude that K_n is Eulerian iff n is odd

e.g.- K_5 is Eulerian graph, as K_5 is connected and the degree of each vertex is $5-1=4$.

Ques.2 Which complete graph $K_{m,n}$ are Euler graphs?

Solⁿ:- The complete Bipartite graph $K_{m,n}$ is Euler graph iff both m and n are even.

e.g.→ In $K_{2,4}$ all the vertices are of even degree (two vertices are of degree 4 and 4 vertices are of degree 2) Hence $K_{2,4}$ is an Euler graph.

Imp

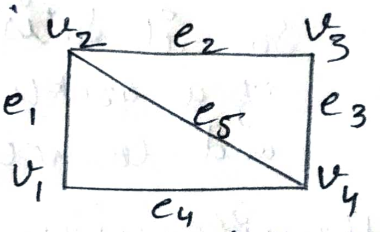
Hamiltonian graph :→

A circuit in a Graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as Hamiltonian cycle.

A Graph G is called Hamiltonian graph if it contains a Hamiltonian cycle.

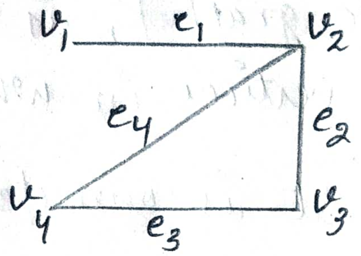
Hamiltonian path is a simple path that contains all the vertices of G exactly once where the end points may be distinct.

e.g.:- Graph G



is a Hamiltonian graph

as it contains a Hamiltonian cycle $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$
It contains all vertices of G but not all edges.

G': 

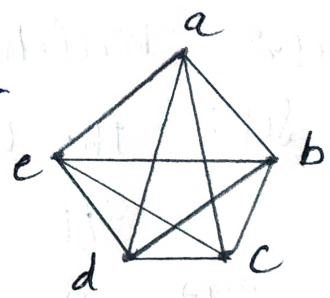
is not a Hamiltonian graph as It does not contain Hamiltonian cycle since every cycle containing every vertex must contain e_1 twice

Dirac's Theorem →

A simple connected graph G with $n \geq 3$ vertices is Hamiltonian if $\deg(v) \geq \frac{n}{2}$ for every vertex v in G.

e.g. :→ complete graph K_5 is Hamiltonian as $\deg(v) = 4 \geq \frac{5}{2}$ for each v in K_5

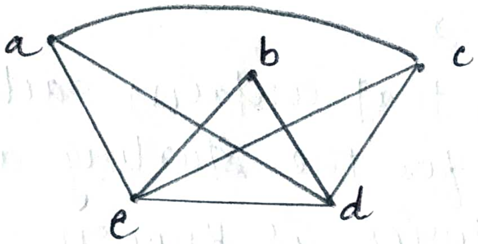
It has Hamiltonian circuit
a - b - c - d - e



K_5

Thm:- A simple connected graph with n vertices and m edges is Hamiltonian if $m \geq \left[\frac{(n-1)(n-2)}{2}\right] + 2$

e.g. :→

G: 

Here no. of vertices $n = 5$
no. of edges $m = 8$

now $\frac{(n-1)\cdot(n-2)}{2} + 2 = \frac{4 \times 3}{2} + 2 = 6 + 2 = 8$

$m \geq \frac{(n-1)\cdot(n-2)}{2} + 2 = 8$

$8 \geq 8$ So it satisfies the condition of a graph which is simple and connected to be Hamiltonian

So Graph G is Hamiltonian.

Ques. Discuss the three classes of graphs C_n, K_n and $K_{m,n}$ for Eulerian and Hamiltonian properties.
construct examples for each one of the following four cases.

i.) an undirected graph that is both Eulerian and Hamiltonian

ii.) an undirected graph that is Eulerian, but not Hamiltonian

iii.) an undirected graph that is Hamiltanian, but not Eulerian.

iv.) an undirected graph that is neither eulerian nor Hamiltanian.

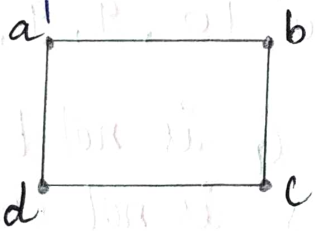
Sol^n:- C_n is Eulerian and Hamiltonian for all $n \geq 3$

K_n is Eulerian iff n is odd
and is Hamiltonian iff $n \geq 3$

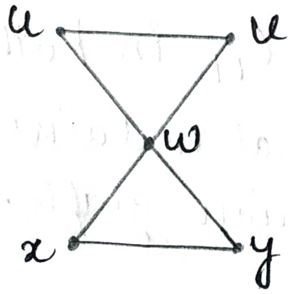
$K_{m,n}$ is Eulerian iff both m and n are even
and is Hamiltonian iff $m = n \geq 2$

i.) Any cycle graph C_n, $n \geq 3$ is both Eulerian & Hamiltonian

C_4:

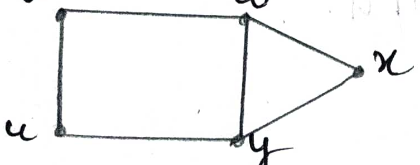


ii.)

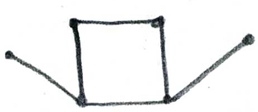


is Eulerian but not Hamiltonian.

iii.)



is Hamiltonian but not Eulerian b/c of having two vertices of odd degree

iv.) $K_{2,3}$:  and  are neither Eulerian nor Hamiltonian graphs.

Ques. Suppose that a connected graph G has 11 vertices and 53 edges. Justify that G is Hamiltonian, but not Eulerian graph.

Imp

OR Suppose G is a (11, 53) type of connected graph. Explain why G is Hamiltonian, but not Eulerian.

Soln:- The complete graph K_{11} has 11 vertices and

$$\frac{11 \cdot (11-1)}{2} = \frac{11 \cdot 10}{2} = 55 \text{ edges}$$

and the graph G has 11 vertices and 53 edges

So we can visualise G as K_{11} with two edges "missing". These two edges may be adjacent edges or non-adjacent edges.

Therefore, the degree sequence of the graph G is given by

$$(10, 10, 10, 10, 10, 10, 10, 10, 9, 9, 8)$$

or $(10, 10, 10, 10, 10, 10, 10, 9, 9, 9, 9)$

Clearly, In both Cases, G is not Eulerian as the degree of all vertices is not even.

However, G contains a cycle C_{11} in both cases,

Result: Every graph of order n that contains the cycle C_n is a Hamiltonian graph.

Hence G is Hamiltonian Graph.

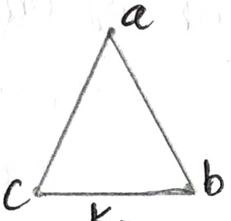
Planar Graph :→

A Graph G is said to be *Planar* if it can be drawn on the plane without any crossover b/w its edges.

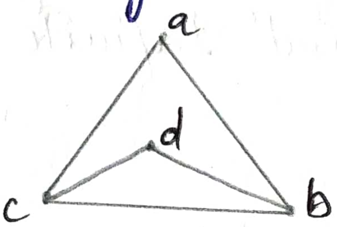
on contrary,

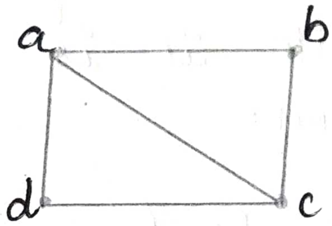
A Graph G that can not be drawn on a plane without a crossover b/w its edges is called Non-planar.

e.g →



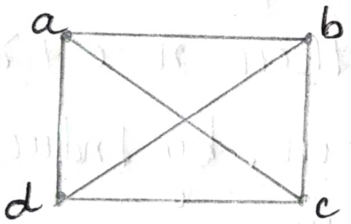
K_3



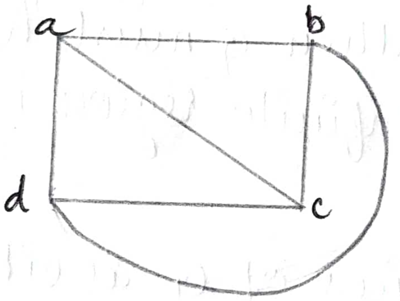


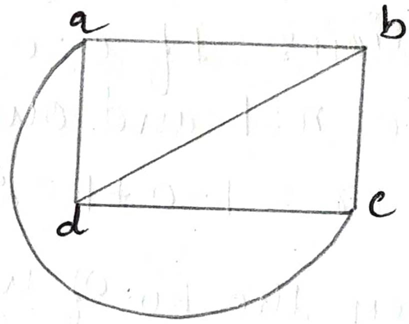
are all planar graphs.

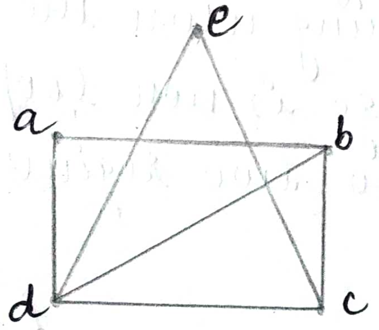
K_4 :



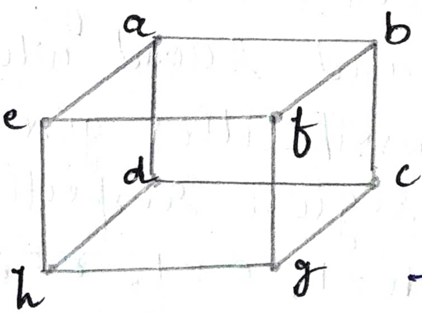
is also a planar graph as it can be drawn on the plane without edges crossing.



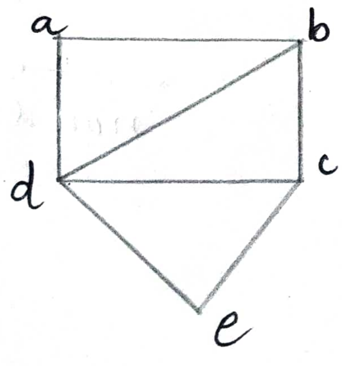


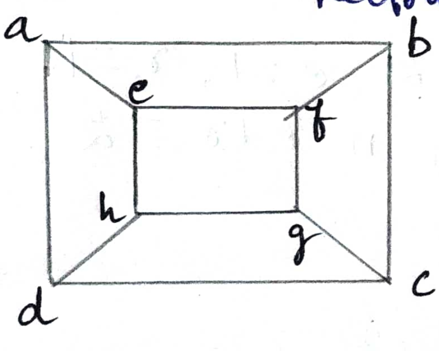


and



are planar graphs as they can be Redrawn as





Region of a planar graph :-

A Region of a planar graph is defined to be an area of the plane that is bounded by edges and is not further divided into subareas.

⇒ if the area of the region is finite then the Region is called a finite region, if the region is Infinite it is called Infinite outer or Unbounded Region.

Imp
Euler's formula :→

statement : If a connected planar graph G has n vertices, e edges and r regions, then $n - e + r = 2$.

Proof :- we will prove this theorem, by Induction method on e, no. of edges in G.

Basis of Induction :- If $e = 0$, then G must have just one vertex i.e $n = 1$ and one Infinite region i.e $r = 1$

Then $n - e + r = 1 - 0 + 1 = 2$

If $e = 1$, then the no. of vertices of G is either 1 or 2, the first possibility of occuring when the edge is a loop and second when edge is non loop These two possibilities give rise to two regions and one region respectively.

for loop, $n = 1$, $e = 1$, $r = 2$

Then $n - e + r = 2$

loop

non-loop, $n = 2$, $e = 1$, $r = 1$

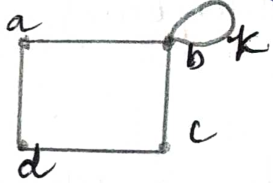
Then $n - e + r = 2$

non loop

Induction hypothesis :- Suppose that the Result is true for any connected planar graph G with e-1 edges let n', e', r' denote the number of vertices, edges and regions in G Then $n' - e' + r' = 2$ —(1)

Induction Step :- Now we add one new edge k to G to form a connected supergraph of G, denoted by G+k, with n, e, and r no. of vertices, edges and regions. Then there are following three possibilities.

i.) if k is a loop, Then a new region bounded by the loop is created but the no. of vertices remains unchanged



In this case, $n - e + r$

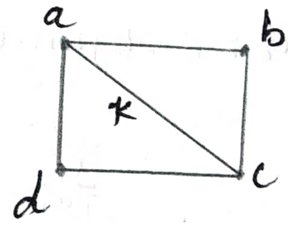
$= n' - (e'+1) + (r'+1)$

$= n' - e' + r'$ from (1)

$= 2$

ii.) if k joins two distinct vertices of G, Then one of the region of G splits into two, so that no. of regions is increased by 1, but no. of vertices remains unchanged

In this case, $n - e + r$

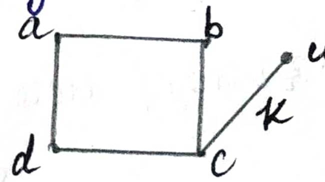


$= n' - (e'+1) + (r'+1)$

$= n' - e' + r'$ from (1)

$= 2$

iii.) if k is incident with only one vertex of G, Then another vertex must be added, increasing the no. of vertices by 1, but leaving the no. of regions unchanged.



In this case, $n - e + r$

$= (n'+1) - (e'+1) + r'$ from (1)

$= n' - e' + r' = 2$

Hence Proved

Result 1:- If G is a connected simple planar graph with $n \geq 3$ vertices and e edges, Then $e \leq 3n-6$.

Imp Ques.1 Show that the complete graph K_5 is non-planar.

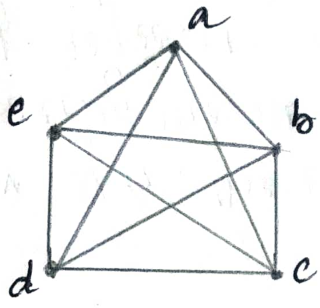
Solⁿ:- In K_5,

no. of vertices $n = 5$

no. of edges $e = \frac{n.(n-1)}{2}$

$= \frac{5.(5-1)}{2}$

$= 10$



K_5

K_5 is connected simple graph, formula: $e \leq 3n-6$

Now $3n-6 = 3 \times 5 - 6 = 9$

So $10 = e \nleq 3n-6 = 9$

Hence K_5 is a non-planar graph

Result:-2 if G is connected simple planar graph with $n \geq 3$ vertices and e edges and no circuits of length 3, Then $e \leq 2n-4$.

Imp Ques.2 Show that complete bipartite graph $K_{3,3}$ (Kuratowski graph) is non-planar.

Solⁿ:- In $K_{3,3}$

no. of vertices $n = 3+3 = 6$

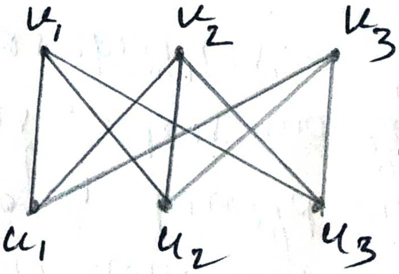
no. of edges $e = 3 \times 3 = 9$

By formula, $e \leq 2n-4$

$2n-4 = 2 \times 6 - 4 = 8$

Since $9 = e \nleq$ [illegible] $-4 = 8$

Hence $K_{3,3}$ is non planar graph.



$K_{3,3}$

$K_{3,3}$ is simple connected graph containing no circuit of length 3.

Ques. Let G be a (6, 12)-type of connected planar graph. Show that every region in G is triangular. Further, give an example of a connected planar graph of order 6 such that $\deg(v) \geq 3$, for all $v \in V(G)$

Solⁿ:- As G is a (6, 12) type of connected planar graph
So no. of vertices $n = 6$

no. of edges $e = 12$

By Euler's formula, $n - e + r = 2$

$$6 - 12 + r = 2 \Rightarrow r = 8$$

So no. of Regions in G, $r = 8$

Now by lemma, In a planar graph G,
Given a region R of G, let e_R be the subset of edges that surround R, then

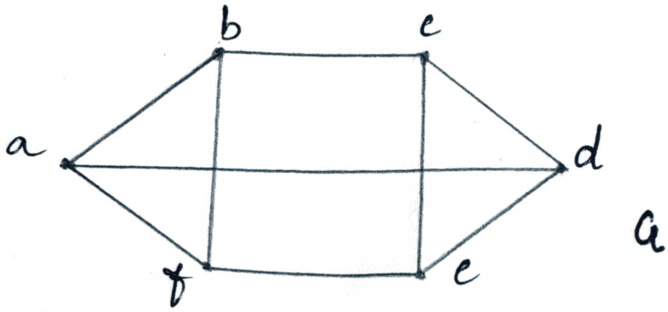
$$\sum_{\text{all regions } R \text{ in } G} |e_R| = 2e$$

$$8|e_R| = 2e = 2 \times 12 = 24$$

$$\Rightarrow |e_R| = \frac{24}{8} = 3$$

Hence each region R in G is surrounded by 3 edges & is triangular region.

A connected planar graph of order 6 in which $\deg(v) \geq 3$ $\forall v \in V(G)$ →



here $\deg(v) = 3$, for all $v \in V(G)$.

Ques. Imp Let a connected planar G has v vertices and $e \geq 3$ edges. Then show that G contains a vertex of deg ≤ 5.

Solⁿ:- We prove it by contradiction,

Suppose the degree of each vertex u in a connected planar graph G is atleast 6. i.e $deg_G(u) \geq 6$, $\forall\ u \in V(G)$.

by Handshaking Thᵐ, $\sum_{u \in V(G)} deg_G(u) = 2e$

$\Rightarrow \quad 6v \leq 2e, \quad \because deg_G(u) \geq 6, \forall\ u \in V(G)$

$\Rightarrow \quad 3v \leq e \quad$ —(1) $\quad$ v is the no. of vertices.

Since G is connected planar graph

So $\quad e \leq 3v - 6 \quad$ —(2)

from (1), (2) we have

$e \leq e - 6 \quad \Rightarrow \quad 0 \leq -6$, which is not possible

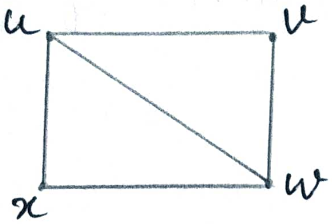
Thus every planar graph has a vertex of degree atleast 5.

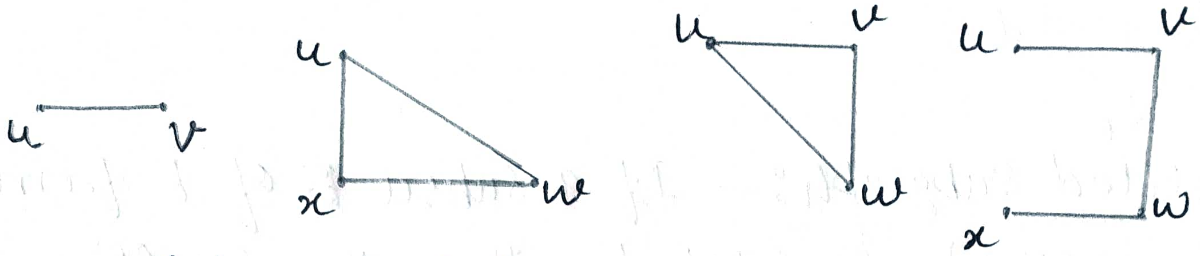
Subgraph :→

Let $G = (V, E)$ be a graph. A graph $H = (V', E')$ is called a Subgraph of G if $V' \subseteq V$ and $E' \subseteq E$

e.g →

G :



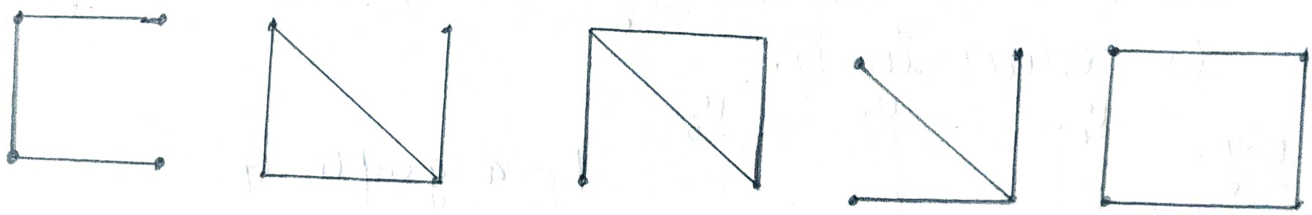


are Subgraphs of G.

Note :- Any Subgraph of a graph G can be obtained by removing certain vertices and edges from G. Removal of an edge leaves its points in place, whereas the removal of a vertex necessitates the removal of any edges that are incident on that vertex.

1. **Spanning Subgraph :-** A subgraph H of a graph G is called a spanning subgraph of G iff $V(H) = V(G)$.

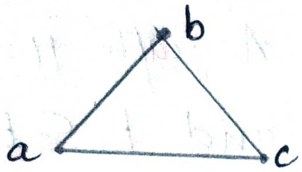
e.g →

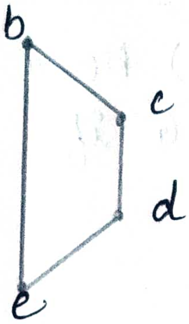


are all spanning subgraphs of G.

2. **vertex deleted Subgraph :-** Let $G(V,E)$ be a graph. If a Subset U of V and all the edges incident on the elements of U are deleted from G, then the Resulting Subgraph is called vertex deleted subgraph of G.

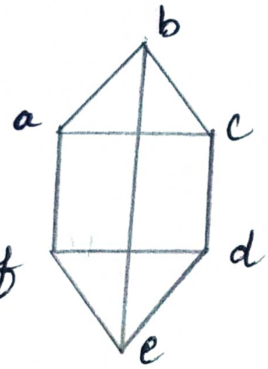
e.g.:-





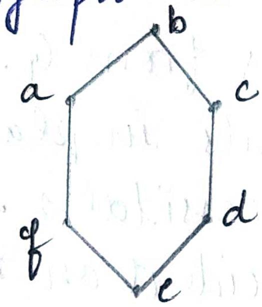
are vertex deleted Subgraphs of G_1

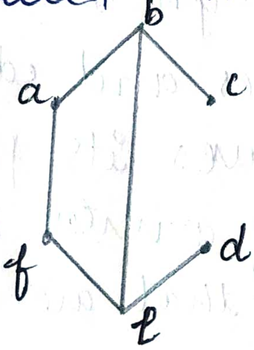
G_1:

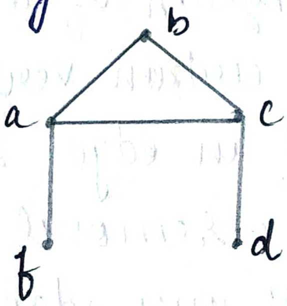


3. **Edge deleted Subgraph:-** If a subset S of E from a graph $G(V,E)$ is deleted, then the resulting Subgraph is called an edge deleted Subgraph of G.

e.g.-



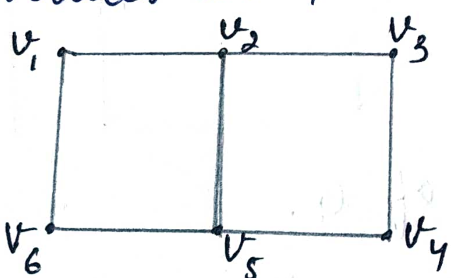




are Edge deleted Subgraphs of G_1.

4. **Induced Subgraph:-** A Subgraph $H(V', E')$ of a Graph $G(V,E)$ is called Induced Subgraph of G if its edge set E' contains all edges in G whose end points belong to vertices in G.

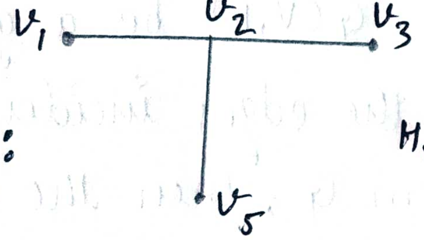
e.g.- Let



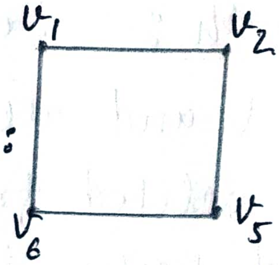
be a graph G

Then

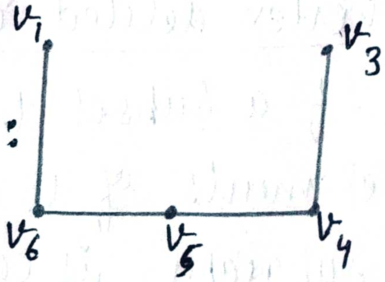
H_1:



H_2:



H_3:



are Induced subgraph of G.

but are not Spanning subgraphs of G as $V(G) = V(H)$.

Graph Isomorphism :→

Two Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be Isomorphic graphs if there exists a one-one and onto function $f : V_1 \to V_2$ such that for every pair of vertices $u, v \in V_1$: $e = (u, v) \in E_1 \Leftrightarrow f = (f(u), f(v)) \in E_2$

In this case, f is called graph Isomorphism b/w Graphs G_1 and G_2.

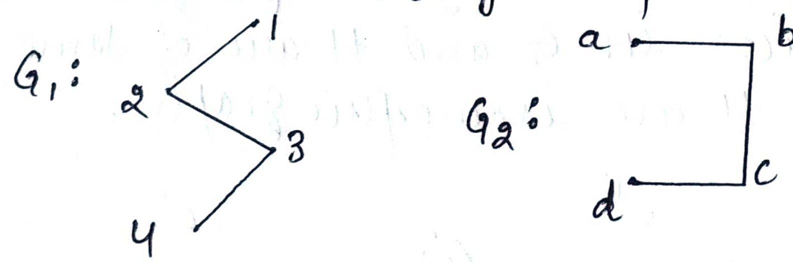
<u>Note</u>:- if $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are Isomorphic graphs then G_1 and G_2 have

i.) Same no. of vertices i.e $|V_1| = |V_2|$

ii.) Same no. of edges i.e $|E_1| = |E_2|$

iii.) Same degree sequence i.e if $deg(v_i) = m$ in G_1 Then $deg(f(v_i))$ must also be m in G_2.

<u>Example:1</u> show that the given pair of graphs are isomorphic.



<u>Soln</u>:- Here $V_1 = \{1, 2, 3, 4\}$, $V_2 = \{a, b, c, d\}$

and $E_1 = \{(1,2), (2,3), (3,4)\}$, $E_2 = \{(a,b), (b,c), (c,d)\}$

$\Rightarrow |V_1| = |V_2|$ and $|E_1| = |E_2|$

The vertices of degree 1 in G_1 are $\{1, 4\}$ and in G_2 are $\{a, d\}$

The vertices of degree 2 in G_1 are $\{2, 3\}$ and in G_2 are $\{b, c\}$

Now define a fⁿ $f : V_1 \to V_2$ as

$f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = d$.

clearly f is one-one & onto.

Further, $(1,2) \in E_1$ and $(f(1), f(2)) = (a,b) \in E_2$

$(2,3) \in E_1$ and $(f(2), f(3)) = (b,c) \in E_2$

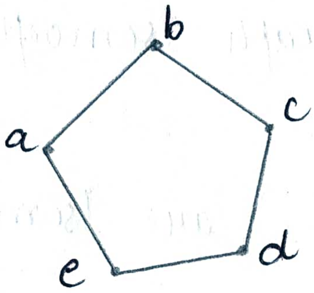
$(3,4) \in E_1$ and $(f(3), f(4)) = (c,d) \in E_2$

Hence f preserves Adjacency of vertices

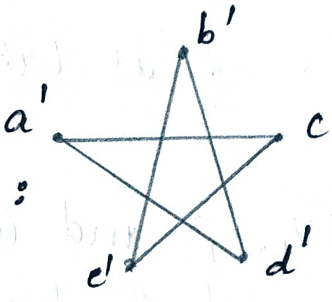
Therefore G_1 and G_2 are isomorphic graphs.

Example. 2

G:



H:



G and H are isomorphic graphs.

as $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$

So $\exists$ a one-one & onto f^n $f: V(G) \to V(H)$

defined as $f(a) = a'$, $f(b) = c'$, $f(c) = e'$,

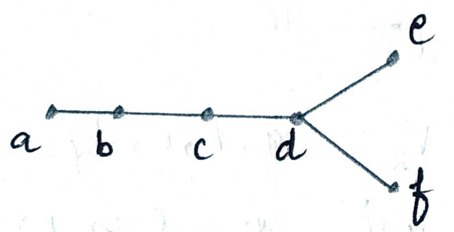
$f(d) = b'$ and $f(e) = d'$.

Such that f preserves adjacency of vertices and all vertices in G and H are of same degree 2

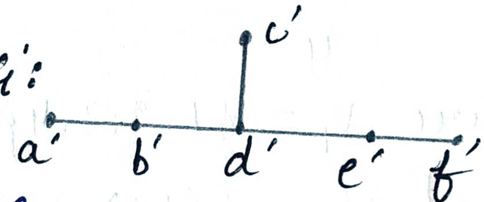
Hence G and H are isomorphic graphs.

Example. 3

G:

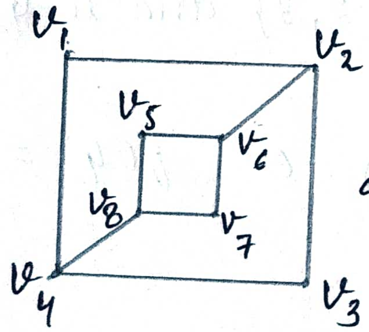


G':

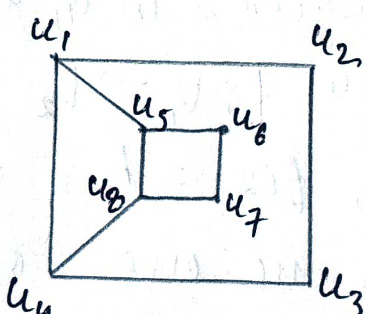


G and G' are not isomorphic graphs as vertex d in G has two pendant vertex whereas its corresponding vertex d' in G' has only one pendant vertex.

Example: 4



and



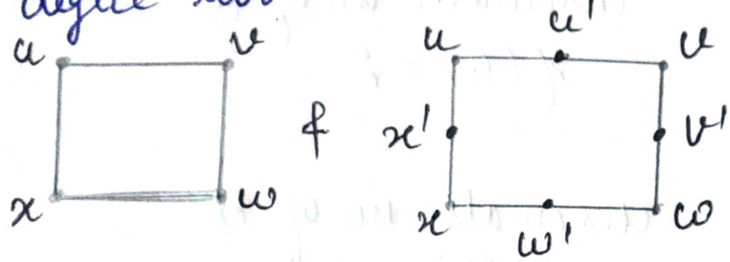
are Not Isomorphic Graphs.

Homeomorphic graphs :→

Two graphs $G(V, E)$ and $G'(V', E')$ are said to be Homeomorphic if one can be obtained from another graph by insertion or deletion of one or more vertices of degree two.

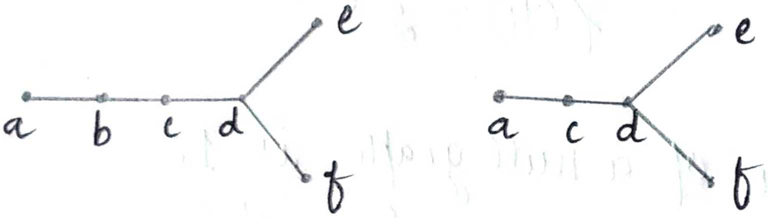
e.g →

i.)



are Homeomorphic graphs.

ii.)



are Homeomorphic graphs.

Graph colouring :→

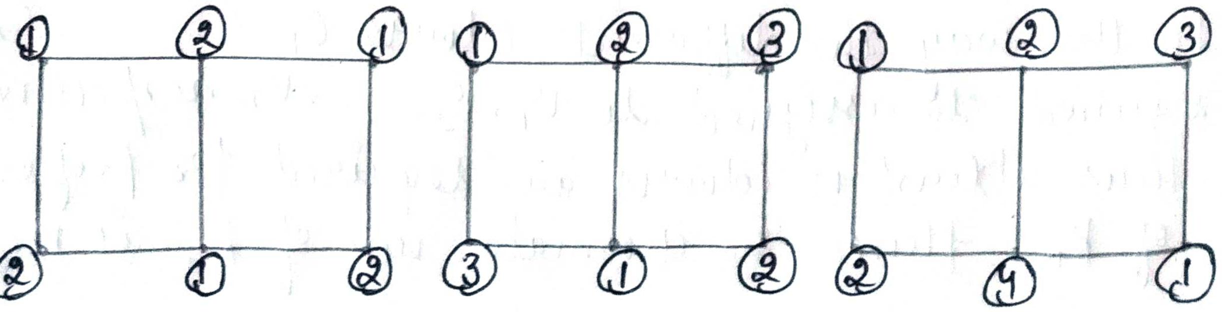
Graph colouring is an assignments of colours to the elements (vertices, edges & regions) of a graph subject to certain constraints.

Vertex colouring :–

The assign of colours to the vertices of a graph G, one colour to each vertex, so that the adjacent vertices are assigned different colours is called the proper colouring or vertex colouring of G.

The n-colouring of G is a proper colouring of G using n-colours. if G has n colouring, then G is said to be n-colourable.

e.g →



2-colourable graph

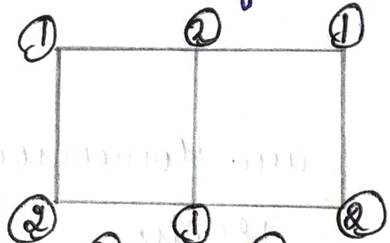
3-colourable graph

4-colourable graph

chromatic Number :- The chromatic number of a Graph G is the minimum number of colours required to assign each vertex of G, such that no two adjacent vertices are of same colour. It is denoted by $\chi(G)$.

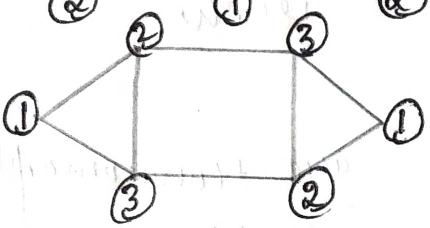
e.g. →

G:



chromatic no. of G
$\chi(G) = 2$

H:



chromatic no. of H
$\chi(H) = 2$

Notes - (1.) The chromatic no. of a null graph is 1.

(2.) $\chi(G) \le n$, n is the no. of vertices of G.

(3.) The chromatic no. of a graph having a triangle is atleast 3.

(4.) if $\deg(v) = d$ for a vertex v in G, then atmost d colours are required for proper colouring of the vertices adjacent to v.

Ex-

(1.) chromatic no. of a complete graph K_n with n vertices $= n$

Let $v_1, v_2, \ldots, v_n$ are n vertices of a complete Graph K_n. Let a colour C_1 be assigned to v_1 and since v_2 is adjacent to v_1, a different colour C_2 is required to assigned to v_2. Since v_3 is adjacent to both v_1 and v_2, so another colour C_3 is required to be assigned to v_3. In this way the different colours $C_1, C_2, \ldots, C_n$ are required to assigned to $v_1, v_2, \ldots, v_n$ respectively. Thus atleast n colours are required for proper colouring of K_n. Hence the chromatic no. of K_n is n.

(2.) chromatic Number of a Bi-partite graph $(K_{m,n}) = 2$

Let G be a Bipartite graph. So its vertex set V can be partitioned in two sets $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$ s.t every edge of G joins some v_i to some u_j. Since no two vertices are adjacent so we can assign a colour-1 to each v_i. Since u_i is adjacent to a v_j and since no two of u_i's are adjacent, so we can assign colour-2 to each u_i. Thus the graph is coloured with only colours. Hence $\chi(G) = 2$.

(3.) chromatic number of a cycle (C_n):-

The chromatic no. of a cycle with n vertices (C_n) is

i.) 2 if n is even

ii.) 3 if n is odd

(4.) chromatic number of a wheel (W_n):-

The chromatic number of a wheel W_n with n vertices is

i.) 3 if n is even

ii.) 4 if n is odd.

Five-colour Theorem :- A Planar graph G is 5-colourable.

OR Every planar graph with n vertices can be coloured using atmost 5 colours.

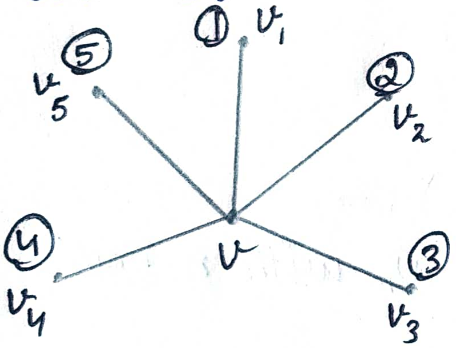
Proof:- we will prove this theorem by Induction on the number n of vertices of G.

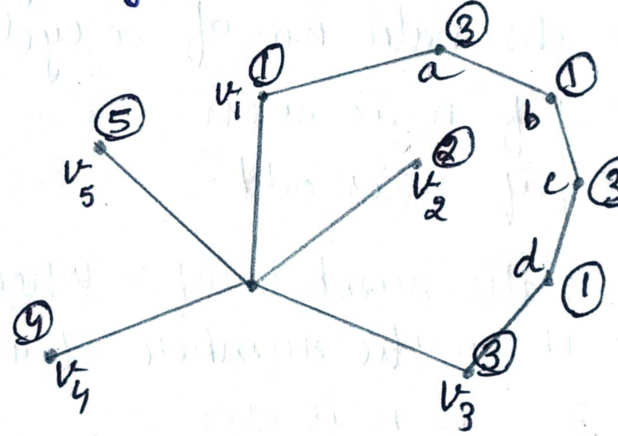
Basis of Induction:- if $n \le 5$, then G can be coloured with ≤ 5 colours.

Induction Hypothesis:- suppose the theorem holds for every planar graph with $n-1$ vertices

Induction Step:-

we shall prove that G is 5-colourable with n vertices.
Since G is a planar connected graph
So it must have a vertex v such that $\deg(v) \le 5$.
Let G' be the graph obtained by deleting v from G.
By Induction Hypothesis, G' requires not more than 5 colours.
if $\deg(v) \le 4$, Then there is no difficulty in Proper colouring, Since we can give to v one more colour
But if the vertex v has degree 5 i.e has exactly five neighbours and they are all differently coloured.





suppose there is a path b/w vertices v_1 and v_3, say $v_1 - a - b - c - d - v_3$ coloured alternatively with colours 1 and 3. Then a Similar path b/w v_2 and v_4 can not exist, since this path, if it exists, will intersect the path b/w v_1 and v_3, contradicting that G is planar. Hence we can Interchange the colours of all vertices connected to v_2, giving colour 4 to v_2, while v_4 has still colour 4. Thus we can colour v with colour 2, and Hence the graph $G' + v = G$ is 5-colourable. ||ly, if we have assumed that no path exists b/w v_1 and v_3, we could colour v_3 with colour 1 and v_1 with colour 3. Thus a planar graph with 5 vertices is 5-colourable.

Edge colouring :-

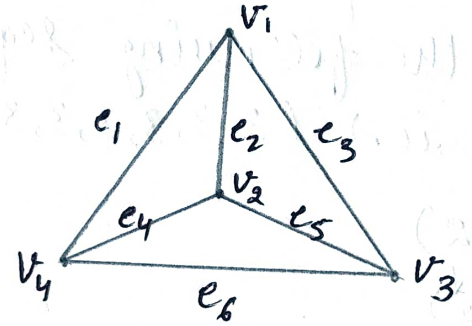
An Edge colouring of a Graph G is an assignment of colours to the edges of G so that no two edges with a common vertex recieve the same colour.

Chromatic Index :-

The minimum no. of colours required in an edge colouring of G is called chromatic Index of G and is denoted by $\chi'(G)$.

Example:- Find the chromatic no. of the Graph

G :



Soln:- we have to assign different colours to the edges having common vertex.

Edge	e_1	e_2	e_3	e_4	e_5	e_6
colour	1	2	3	3	1	2

only three colours are required for Edge colouring Therefore $\chi'(G) = 3$.

Note:- If G has K-edge colouring, then G is said to be K edge colourable.

Combinatorics

Combinatorial problems deals with selections and arrangements of objects with or without repetitions.

→ The Problems related to selections are about combinations denoted by ${}^nC_r$ or $\binom{n}{r}$ the no. of combination of n distinct objects taken r at a time (without repetition)

→ The problems related to arrangements are about Permutations denoted by $P(n, r)$ no. of permutations of n distinct objects taken r at a time (without repetition)

2023

Ex:-1 How many triangles can be formed by joining 10 points such that 5 of these lie on the same line.

Solⁿ:- we need three non-collinear points to make a triangle. Therefore, from 10 such points, we can make ${}^{10}C_3 = 120$ number of triangles. However, 5 of these 10 points are collinear, we can make only ${}^{10}C_3 - {}^5C_3 = 120 - 10 = 110$

number of triangles.

Ex:-2 Three prizes are to be awarded among 10 candidates. In How many ways can the prizes be given, so that no candidate may get more than one prize?

Solⁿ:- Since no candidate get more than one prize, the required number is equal to the number of permutations of 10 different things taken 3 at a time i.e,

$$P(10,3) = {}^{10}P_3 = \frac{10!}{7!} = 10 \times 9 \times 8 = \underline{\underline{720}}$$

Counting Principles :-

(1.) **Sum Rule :-** If an event can occur in m-ways and another event can occur in n-ways, and if these two events can not occur simultaneously, then one of the two events can occur in $m+n$ ways.

In General if E_i $(i=1, 2, \ldots, k)$ are k events such that no two of them can occur at same, and if E_i can occur in n_i ways, Then one of the k events can occur in $n_1+n_2+n_3+ \ldots +n_k$ ways.

e.g → If a student is getting admission in 4-different Engineering colleges and 5 medical colleges, find the number of ways of choosing one of the above college.

Soln:- using sum rule, there are $4+5=9$ ways of choosing one of the colleges.

(2.) **Product Rule :-** If an event can occur in m-ways and a second event in n-ways and if the number of ways the second event occurs, does not depend on the occurrence of the first event, Then the two can occur simultaneously in $m \times n$ ways.

In General, if E_i $(i=1, 2, \ldots, k)$ are k events and if E_1 can occur in n_1 ways, E_2 can occur in n_2 ways (no matter E_1 occurs) E_3 can occur in n_3 ways (no matter E_1 and E_2 occurs) ..., E_k occurs in n_k ways (no matter how $k-1$ events occurs) Then k events can occur simultaneously in $n_1 \times n_2 \times n_3 \times \ldots \times n_k$ ways.

eg → Three persons enter into car, where there are 5 seats. In how many ways can they take up their seats?

Sol^n:- first person has a choice of 5 seats and can sit in any one of these 5 seats. So there are 5-ways of occupying the first seat. The second person has a choice of 4 seats. Similarly, the third person has a choice of 3 seats. Hence, the required no. of ways in which all the three persons can sit is $5 \times 4 \times 3 = 60$.

Problems related to Integer solutions :→

→ The Number of Integer solutions of the equation $x_1 + x_2 + \dots + x_n = r$ subject to $x_i \geq 0 \;\forall\; 1 \leq i \leq n$ is ${}^{n+r-1}C_r \left(= {}^{n+r-1}C_{n-1}\right)$

→ The Number of Positive Integer Solutions of eq^n $x_1 + x_2 + \dots + x_n = r$, where $x_1 \geq 1, x_2 \geq 1, \dots, x_n \geq 1$ is the same as the no. of ways to distributive n identical things among r persons $= {}^{r}C_{n-1}$

Ex:-1 How many Integer solutions does the eq^n $x+y+z=17$ have, where x, y, z are non negative integers?

Sol^n:- given eq^n is $x+y+z=17$, $x, y, z \geq 0$

Here, $n=3$, $r=17$

So No. of Integer sol^n of given eq^n

$$= {}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$$

$$= {}^{17+3-1}C_{3-1} = {}^{19}C_2 = \frac{19 \times 18}{2} = 171$$

Ex:-2 find the Number of Integer Solutions of the equation $x_1 + x_2 + x_3 + x_4 = 32$, Subject to $x_i \geq 0$ for all $1 \leq i \leq 4$.

Solⁿ:- Here $n = 4$, $r = 32$

No. of Integers Solutions of given eqⁿ

$$= {}^{n+r-1}C_r$$

$$= {}^{4+32-1}C_{32}$$

$$= {}^{35}C_{32} = \underline{6545}.$$

Imp.

Ques.1 find the Number of Integer solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 30$, subject to constraints $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 4$, $x_4 \geq 2$, $x_5 \geq 0$.

Solⁿ:- taking $u_1 = x_1 - 2$, $u_2 = x_2 - 3$, $u_3 = x_3 - 4$, $u_4 = x_4 - 2$ and $u_5 = x_5$

So that $u_1 + 2 = x_1$, $u_2 + 3 = x_2$, $u_3 + 4 = x_3$, $u_4 + 2 = x_4$ and $u_5 = x_5$ becomes non-negative Integers.

Therefore the given asks to find the no. of Integer solⁿ of the equation,

$$(u_1+2) + (u_2+3) + (u_3+4) + (u_4+2) + u_5 = 30$$

$$\Rightarrow u_1 + u_2 + u_3 + u_4 + u_5 = 19, \quad u_i \geq 0 \ \forall \ 1 \leq i \leq 5$$

Hence Required No. of Integer solutions

$$= {}^{n+r-1}C_r$$

$$= {}^{5+19-1}C_{19}$$

$$= {}^{23}C_{19}$$

$$= {}^{23}C_4 = \underline{8855}.$$

Imp

Inclusion – Exclusion Principle :→

if $A_1, A_2, \ldots, A_n$ are n sets with universal Set U. Then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$
$$- \ldots\ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

and $\boxed{|A_1' \cap A_2' \cap \ldots \cap A_n'| = |U| - |A_1 \cup A_2 \cup \ldots \cup A_n|}$

In particular,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

2019

Q.1 In a class of 2092 students, 1232 are pursuing a course in spanish; 879 in french; and 114 in Russian. Further, 103 are pursuing courses in both spanish and French; 23 in both spanish and Russian; and 14 in both french and Russian. Assuming that each student is pursuing at least one of three languages courses, find how many of them are pursuing a course in all the three languages.

Solⁿ:- Let S, F and R Respectively denote the set of students who are pursuing a course in spanish, French and Russian.

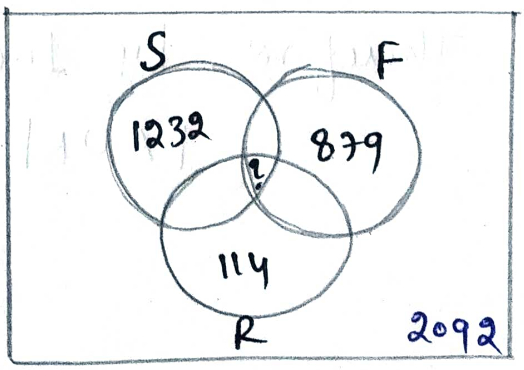
Then we have,

$|S| = 1232$, $|F| = 879$

$|R| = 114$, $|S \cap F| = 103$

$|S \cap R| = 23$, $|F \cap R| = 14$

$|S \cup F \cup R| = 2092$



we have to find $|S \cap F \cap R|$

from Inclusion exclusion principle, we have

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

$$\Rightarrow 2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$$

$$\Rightarrow |S \cap F \cap R| = 2092 - 2085$$

$$= \underline{7} \text{ Ans}$$

Ques. 2 How many bit strings of length 8 start with a 1 bit or end with two bits 00 ?

Solⁿ:- Let A be the set of 8-bit strings that start with a bit 1, and B be the set of 8-bit strings that end with two bits 00.
Then $A \cup B$ contains 8-bit strings that starts with a 1 bit or end with two bits 00, and $A \cap B$ is the set of 8-bit strings that begin with a 1 bit and ends with two bits 00.
By the product rule, since the first bit can be chosen in only one way and each of the remaining 7 bits can be chosen in two ways.

So $|A| = 2^7 = 128$ $\underline{1}\,\_\,\_\,\_\,\_\,\_\,\_\,\_$

Similarly, $|B| = 2^6 = 64$ $\_\,\_\,\_\,\_\,\_\,\_\,\underline{0}\,\underline{0}$

and $|A \cap B| = 2^5 = 32$ $\underline{1}\,\_\,\_\,\_\,\_\,\_\,\underline{0}\,\underline{0}$

Therefore, by Inclusion Exclusion principle

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 128 + 64 - 32$$

$$= \underline{160} \text{ Ans}$$

Imp

Ques. 3 Find the number of 10-bit strings that starts with three 0's or end with two 1's.

Solⁿ:- Let A denotes the set of 10-bit strings that begin with three 0's, and

B denotes the set of 10-bit strings that end with two 1's.

Then $A \cup B$ contains 10-bit strings that starts with 000 or end with 11.

and $A \cap B$ contains 10-bit strings that starts with 000 and end with 11.

Then By Product rule we have,

$|A| = 2^7$ 0 0 0 _ _ _ _ _ _ _

$|B| = 2^8$ _ _ _ _ _ _ _ _ 1 1

$|A \cap B| = 2^5$ 0 0 0 _ _ _ _ _ 1 1

By Inclusion Exclusion principle,

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 2^7 + 2^8 - 2^5$$
$$= 128 + 256 - 32 = \underline{352} \quad \text{Ans}$$

Ques. 4. How many Integers between 1000 and 9999 are divisible by 7 or 11.

Solⁿ:- Let A and B be respectively the sets of Integers b/w 1000 and 9999 divisible by 7 and 11. Then

$$|A| = \left\lfloor \frac{9999}{7} \right\rfloor - \left\lfloor \frac{1000}{7} \right\rfloor = 1428 - 142 = 1286$$

$$|B| = \left\lfloor \frac{9999}{11} \right\rfloor - \left\lfloor \frac{1000}{11} \right\rfloor = 909 - 90 = 819$$

Similarly, ∵ 7 and 11 are relatively prime, we have

$$|A \cap B| = \left\lfloor \frac{9999}{77} \right\rfloor - \left\lfloor \frac{1000}{77} \right\rfloor = 129 - 12 = 117$$

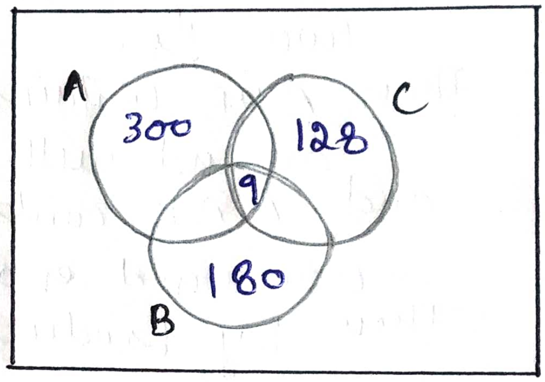
Hence

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 1286 + 816 - 117 = \underline{1985}$$

Imp

Ques.8 find the Number of Integers between 100 and 1000 that are divisible by 3, 5 or 7.

Sol^n:- Let A, B and C denote the Sets of Integers b/w 100 and 1000 that are divisible by 3, 5 and 7 respectively. Then we have



$$|A| = \left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{100}{3} \right\rfloor$$

$$= 333 - 33 = 300$$

$$|B| = \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{100}{5} \right\rfloor$$

$$= 200 - 20$$

$$= 180$$

$$|C| = \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{100}{7} \right\rfloor = 142 - 14 = 128$$

Also, since 3, 5 and 7 are relatively prime Integers, we have

$$|A \cap B| = \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{100}{15} \right\rfloor = 66 - 6 = 60$$

$$|A \cap C| = \left\lfloor \frac{1000}{21} \right\rfloor - \left\lfloor \frac{100}{21} \right\rfloor = 47 - 4 = 43$$

$$|B \cap C| = \left\lfloor \frac{1000}{35} \right\rfloor - \left\lfloor \frac{100}{35} \right\rfloor = 28 - 2 = 26$$

$$|A \cap B \cap C| = \left\lfloor \frac{1000}{105} \right\rfloor - \left\lfloor \frac{100}{105} \right\rfloor = 9 - 0 = 9$$

Hence from Inclusion-exclusion principle, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 300 + 180 + 128 - 60 - 43 - 26 + 9$$

$$= 608 - 129 + 9$$

$$= 488$$

Ques. 6 How many Integers between 1 and 500 are not divisible by 2, 3, 5 or 7.

Solⁿ:- Let U be the set of Positive Integers not exceeding 500. and let A, B, C and D be respectively the set of Integers between 1 and 500 that are divisible by 2, 3, 5 and 7. Thus we have

$$|A| = \left\lfloor \frac{500}{2} \right\rfloor = 250, \quad |B| = \left\lfloor \frac{500}{3} \right\rfloor = 166$$

$$|C| = \left\lfloor \frac{500}{5} \right\rfloor = 100, \quad |D| = \left\lfloor \frac{500}{7} \right\rfloor = 71$$

By Inclusion-Exclusion principle,

$$|A \cup B \cup C \cup D| = \{|A| + |B| + |C| + |D|\} - \{|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|\} + \{|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|\} - \{|A \cap B \cap C \cap D|\} \quad \text{--- (1)}$$

Since 2, 3, 5 and 7 are relatively prime.

So

$$|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|$$

$$= \left\lfloor \frac{500}{6} \right\rfloor + \left\lfloor \frac{500}{10} \right\rfloor + \left\lfloor \frac{500}{14} \right\rfloor + \left\lfloor \frac{500}{15} \right\rfloor + \left\lfloor \frac{500}{21} \right\rfloor + \left\lfloor \frac{500}{35} \right\rfloor$$

$$= 83 + 50 + 35 + 33 + 23 + 14$$

$$= 238$$

$$|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$

$$= \left\lfloor \frac{500}{30} \right\rfloor + \left\lfloor \frac{50}{42} \right\rfloor + \left\lfloor \frac{500}{70} \right\rfloor + \left\lfloor \frac{500}{105} \right\rfloor$$

$$= 16 + 11 + 7 + 4 = 38$$

and $|A \cap B \cap C \cap D| = \left\lfloor \frac{500}{210} \right\rfloor = 2$

Putting all these values in eqⁿ ①, we get the No. of Integers b/w 1 and 500 that are divisible by 2, 3, 5 or 7

$$|A \cup B \cup C \cup D| = 587 - 238 + 38 - 2$$

$$= 385$$

Hence the No. of Integers b/w 1 and 500 that are not divisible by 2, 3, 5 or 7 is given by

$$|A' \cap B' \cap C' \cap D'| = U - |A \cup B \cup C \cup D|$$

$$= 500 - 385$$

$$= \underline{\underline{115}}$$

Imp.

Pigeon-Hole Principle :→

If K+1 or more pigeons are to be accommodated in K pigeonholes, then atleast one pigeonhole must contain atleast two pigeons.

OR

if n objects are to be placed in m boxes, with n > m then atleast one box must contain two or more objects.

eg → Since there are only 365 days in a non-leap year, among 366 people, atleast two must share their birthday.

Since there are only 26 English alphabets, among 27 English words, atleast two must have the same first letter.

Ex:- if 6 colours are used to paint 37 home. show that atleast 7 home of them will be of same colour.

Sol^n:-

$$\left\lfloor \frac{37}{6} \right\rfloor = 6$$

$\Rightarrow$ 6 home each of 6 colour

But one left home will be of the same colour from 6

$\Rightarrow$ 7 home may have a same cycle.

④ Generalized Pigeonhole Principle :→

If $nk+1$ objects are to be placed into n boxes, Then atleast one box contains more than k objects.

OR

If n objects are to be placed in m boxes, with $n > m$ then one box must contain atleast $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ objects.

where $\lfloor\ \rfloor$ denotes greatest Integer funct^n.

Proof:- suppose each box contains atmost $k = \left\lfloor \frac{n-1}{m} \right\rfloor$ objects, then the maximum number of objects accommodated in m boxes is given by

$$mk = m\left\lfloor \frac{n-1}{m} \right\rfloor \le m \cdot \frac{n-1}{m} = n-1$$

which is not the case.

Alternatively, by division algorithm, we can find Positive Integers q, r such that

$$m = qn + r, \qquad 0 \le r < n$$

suppose no box contains $\lceil \frac{m}{n} \rceil$ objects, and let us consider two cases:

1. when $r = 0$ so that $\lceil \frac{m}{n} \rceil = \frac{m}{n} = q$ we have that every box contains less than q objects and hence there are less than $nq = m$ objects in total in n boxes, which is a contradiction.

2. When $1 \le r < n$ so that $\lceil \frac{m}{n} \rceil = q+1$, every box contains at most q objects, and hence $nq = m - r < m$ objects in total which again give a contradiction.

Ex:-1 Among 25 students in a class, how many of them were born in the same month. also on the same day of a week.

Sol^n:-

Since there are only 12 months in an year
So in this case, we have $m = 12$, $n = 25$

$\therefore$ atleast $\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{25-1}{12} \right\rfloor + 1 = 3$ were born in the same month.

Since there are only 7 days in a week
So In this case, we have $m = 7$ and $n = 25$

$\therefore$ atleast $\left\lfloor \frac{25-1}{7} \right\rfloor + 1 = 4$ were born the same day of a week.

Ex:-2. Suppose 45 time slots are available to prepare a time table for 500 classes. How many classrooms are needed to do so.

Sol^n:- We have to prepare a time-table for $n = 500$ classes to be conducted in $m = 45$ slots.
Therefore, by the generalized pigeonhole principle, we have

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{500-1}{45} \right\rfloor + 1 = 12$$

Hence, the least no. of classrooms needed is 12.

Ques.1 Use pigeonhole principle to justify that, among six people in a party, atleast three are mutually friends or atleast three are mutually strangers.

Sol^n:- Suppose 'Aman' is one of six people. we consider two disjoint Sets consisting of "friends of Aman" and 'stranger to Aman'. we do have names of these remaining five people.

So By Generalized pigeonhole principle,

there are atleast $\left\lfloor \frac{5-1}{2} \right\rfloor + 1 = 3$ names in one of the two Sets.

Ques.2. Use principle of pigeonhole, justify that, among 11 numbers selected randomly between 1 and 330, there are atleast two numbers such that their difference is less than or equal to 38.